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Economics Working Paper 15105

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March 28, 2015

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Abstract

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JEL classification : C61, C63, C68, E31, E52

Key Words : nonstationary models, unbalanced growth, time varying transition probabilities, time varying parameters, anticipated shock, shooting method, parameter shift, parameter drift, regime switch, stochastic volatility, capital augmenting, seasonality, Fair and Taylor, extended path, Smolyak method

*An earlier version of this paper circulated under the title "A Tractable Framework for Analyzing Nonstationary and Unbalanced Growth Models". We thank for valuable comments Jesús Fernández-Villaverde, Kenneth Judd, Pablo Kurlat, Monika Piazzesi, Thomas Sargent, Martin Schneider, as well as other seminar participants at Stanford University and Santa Clara University, and graduate students of Econ-288 Computational Economics course at Stanford University.

1 Introduction

We study a class of infinite-horizon nonlinear dynamic economic models in which preferences, technology and laws of motion for exogenous variables can change over time either deterministically or stochastically, according to a Markov process, or both. A distinctive feature of our analysis is that we allow for Markov processes with time-varying transition probabilities. Unbalanced stochastic growth models fit into that class, but so do many other models and applications such as the entry into a monetary union, a nonrecurrent policy regime switch or deterministic seasonals. The studied models are nonstationary in the sense that the decision and value functions are time-dependent and cannot be generally solved with conventional numerical methods that construct stationary Markov equilibria.

Two clarifying comments are in order: First, some nonstationary models from the studied class can be converted into stationary, for example, a nonstationary model with a balanced growth path can be converted into stationary by using an appropriate change of variables. We do not focus on those special cases but on a generic nonstationary model for which a stationary representation is unknown. Second, Markov processes can be nonstationary even if all the parameters and transition probabilities are time-invariant, for example, unit root and explosive processes are nonstationary. The latter kind of nonstationary processes in not explicitly studied in the present paper.

We introduce a quantitative framework, called *extended function path* (EFP), for calibrating, solving, simulating and estimating the studied class of nonstationary Markov models. EFP is aimed to accurately approximate time-varying decision functions in a nonstationary economy during a given number of periods τ. It assumes that in some remote period $T \gg \tau$, the economy becomes stationary, and it proceeds in two steps: First, it constructs conventional stationary Markov decision functions for the stationary economy for $t \geq T$; and then, it finds a path of time-varying decision functions for periods $t = 0, ...T-1$ that matches the given T-period terminal condition (i.e., the stationary Markov decision functions for $t \geq T$). If T is sufficiently large relatively to τ, the EFP approximation in the first τ periods is not sensitive to the specific terminal condition used. Therefore, we obtain an accurate approximation for time-varying decision and value functions in the first τ periods (the remaining decision functions for the periods $T - \tau$ are discarded).

EFP resembles solution methods for finite-horizon models with a given terminal condition (e.g., life-cycle models), however, for the studied class of infinite-horizon problems, there is no terminal condition any finite period. Hence, EFP constructs a finite-horizon approximation to an infinite-horizon nonstationary problem by using an appropriate truncation.

We develop theoretical foundations for the EFP framework in the context of the constructed class of nonstationary Markov models. First, we provide a set of assumptions under which the optimal decision and value functions in the T-period stationary economy are state-contingent, i.e., memoryless concerning a specific history that leads to the current state. In our case, time-dependency takes a particular tractable form for the model's endogenous variables, namely, the optimal decision and value functions follow a Markov process with possibly time-varying transition probabilities. Second, we prove a turnpike theorem that shows that a solution to the T-period stationary model converges to the true solution of the nonstationary models as T increases. This result implies that EFP is capable of approximating a solution to a nonstationary infinite-horizon problems with an arbitrary degree of precision.

Another method in the literature that can solve nonstationary infinite-horizon Markov models is the extended path (EP) framework of Fair and Taylor (1983). To deal with uncertainty, Fair and Taylor (1983) propose to use the certainty-equivalence approach, namely, they replace expectation of a function across states by a value of the function in the expected state. The EP and EFP methods are similar in that they both extend the path, i.e., they both construct an approximate solution for larger time horizon T than time horizon τ for which the solution is actually needed (to mitigate the effect of an arbitrary terminal condition on the approximation during the initial τ periods). However, the two methods differ critically in the object they construct and in the way they approximate expectation functions, namely, Fair and Taylor's (1983) method constructs a path for variables by using the certainty equivalence approach, while EFP constructs a path for decision functions by using more accurate integration techniques such as Monte Carlo, quadrature and monomial ones. As a result, EP can accurately solve linear models with uncertainty (in such models, the certainty equivalence approach leads to exact approximation of integrals), as well as nonlinear models without uncertainty, whereas EFP can also accurately solve nonlinear models in which the certainty equivalence approach is either not applicable or leads to inaccurate solutions.

Although numerical examples in the paper are limited to models with two or three state variables, we design EFP in the way that makes it tractable in large-scale applications. A specific combination of computational techniques that we use includes Smolyak sparse grids (see, e.g., Krueger and Kubler (2004) and Judd, Maliar, Maliar and Valero (2014)), nonproduct monomial integration methods (such methods are inexpensive and produce more accurate approximations to integrals than Monte Carlo methods, see Judd, Maliar and Maliar (2011) for comparison results) and derivative-free solvers (we use Gauss-Jacobi iteration in line with Fair and Taylor (1983)); see Maliar and Maliar (2014) for surveys of these and other computational techniques that are tractable in problems with high dimensionality (up to 100 state variables). Examples of MATLAB code are provided in webpages of the authors.

Our numerical analysis includes two parts. First, we assess the performance of EFP in a nonstationary test model with a balanced growth path for which a high-quality approximation is available, and we find EFP to be both accurate and reliable. Then, we apply EFP to analyze a collection of challenging nonstationary and unbalanced growth applications that do not admit conventional stationary Markov equilibria. These applications are discussed below.

Capital augmenting technological progress. Acemoglu (2002) argues that technical change may be directed toward different factors of production; and Acemoglu (2003) explicitly incorporates capital augmenting technological progress into a deterministic model of endogenous technological change. However, the assumption of capital augmenting technological progress is inconsistent with a balanced growth path in the standard stochastic growth model but only is the assumption of labor augmenting technological progress; see King, Plosser and Rebello (1988). In our first application, we use EFP to solve a unbalanced growth model with capital augmenting technological progress that does not admit a stationary Markov equilibrium. Our numerical results show that business cycle fluctuations are similar in the models with capital and labor augmenting technological progresses, however, in the former model, the growth rate of capital declines over time while in the latter model, it is constant.

Anticipated versus unanticipated regime switches. The literature on regime switches focuses on unanticipated recurring regime switches (parameter shifts); see Sims and Zha (2006), Davig and Leeper (2007, 2009), Farmer, Waggoner, and Zha (2011), Foerster, Rubio-Ramírez, Waggoner and Zha (2013) and Zhong (2015), among others. However, there are regime switches that are anticipated nonrecurrent, e.g., presidential elections with predictable outcomes, policy announcements, anticipated legislative changes. The idea that anticipated shocks are important for economic fluctuations is dated back to Pigou (1927) and is advocated in, e.g., Cochrane (1994), Beaudry and Portier (2006), Schmitt-Grohé and Uribe (2012). An announcement of accession of new members to the European Union produced important anticipatory effects; see Garmel, Maliar and Maliar (2008). In our second application, we use EFP to construct a nonstationary Markov solution of a growth model that experiences a combination of anticipated and unanticipated technological changes. Our analysis reveals important anticipatory effects relatively to naive solutions in which anticipated switches in regimes are ignored.

Seasonal changes. Seasonal adjustments are a special case of anticipated regime switches; see Barsky and Miron (1989) for well documented evidence on the importance of seasonal changes for the business cycle. Saijo (2013) argues that inadequate treatment of seasonal changes may lead to a significant bias in the parameter estimates. Two approaches are available in the literature to study models with seasonal changes: first, Hansen and Sargent (1993, 2013) use spectral density of variables to construct periodic optimal decision rules; and second, Christiano and Todd (2002) linearize the model around a seasonally-varying steady state growth path and solve for four distinct decision rules. EFP provides an alternative simple and general framework for analyzing seasonal variations. As an example, we construct a nonstationary Markov solution to a growth model with periodic anticipated seasonal changes, and we find a dramatic smoothing effect of seasonal changes on the model's endogenous variables.

Parameter drifting. There is ample evidence in favor of parameter drifting in economic models; see, e.g., Clarida, Galí and Gertler (2000), Lubick and Schorfheide (2004), Cogley and Sargent (2005), Goodfriend and King (2009), and Canova (2009). Furthermore, Galí (2006) argues that nonrecurrent regime changes with gradual policy variations are empirically relevant. However, parameter drifting is generally inconsistent with Markov equilibrium because decision functions gradually change (drift) over time. In our third application, we use EFP to construct a nonstationary Markov solution of a stochastic growth model with parameter drifting. We again observe important anticipatory effects relative to naive solutions in which anticipated parameter drifting is ignored.

Stochastic volatility versus deterministic trend in volatility. A large body of recent literature documents the importance of degrees of uncertainty for the business cycle behavior; see, e.g., Bloom (2009), Fernández-Villaverde and Rubio-Ramírez (2010), Fernández-Villaverde, Guerrón-Quintana and Rubio-Ramírez (2010). The literature normally assumes that the standard deviation of exogenous shocks either follows a Markov process or experiences recurring Markov regime switches. However, there is empirical evidence that volatility of output has a well pronounced time trend, for example, see Mc Connel and Pérez-Quiros (2000), Blanchard and Simon (2001) and Stock and Watson (2003). In our experiment, we construct a nonstationary Markov solution of a growth model in which volatility of shocks gradually decreases over time, as suggested by the analysis of Mc Connel and Pérez-Quiros (2000). As expected, volatility of endogenous variables in our model gradually decreases over time in response to decreasing volatility of shocks.

Calibration and estimation of parameters in nonstationary and unbalanced growth models. There are econometric methods that estimate and calibrate parameters in economic models by constructing and simulating numerical solutions, including simulated method of moments (e.g., Canova (2007)); Bayesian analysis (e.g., Smets and Wouters (2003), and Del Negro, Schorfheide, Smets and Wouters (2007)); and maximum likelihood method (e.g., Fernández-Villaverde and Rubio-Ramírez (2007)). In the fourth application, we illustrate how EFP can be used to calibrate and estimate parameters in an unbalanced growth model by constructing and simulating nonstationary Markov solutions. We specifically consider a model in which the depreciation rate of capital has both a deterministic time trend and a stochastic cyclical component. Shocks to the depreciation rate are introduced in, e.g., Liu, Waggoner and Zha (2011), Gourio (2012) and Zhong (2015); also, see Karabarbounis and Neiman (2014) for evidence on the evolution of the depreciation rate over time. We simulate a time series solution using the fitted parameter values, and we obtain unbalanced growth patterns that closely resemble those observed in the data on the U.S. economy.

The proposed EFP framework is related to several streams of economic literature. First, the theoretical foundations of EFP build on early theoretical literature that studies stochastic growth models with deterministically time-varying utility and production functions; see Majumdar and Zilcha (1987), Mitra and Nyarko (1991) and Joshi (1997).[1] Our main contributions to that literature is that we propose a practical approach to constructing numerical solutions of an empirically-relevant class of nonstationary Markov models, whereas the previous literature was limited to purely theoretical existence results.

Second, EFP is related to conventional numerical techniques for constructing stationary Markov decision functions. Although such techniques cannot be used for solving the studied class of nonstationary models, they can be used as ingredients of EFP, including projection techniques (see, e.g., Judd (1992), Christiano and Fisher (2000), Maliar and Maliar (2015)); perturbation techniques (see, e.g., Judd and Guu (1993), Collard and Juillard (2001), Schmitt-Grohé and Uribe (2004)); stochastic simulation techniques (see, e.g., Den Haan and Marcet (1990), Judd, Maliar and Maliar (2011)); and numerical dynamic programming techniques, in particular, those designed to deal with large scale applications (see, e.g., Smith (1993), Rust (1996), Carroll (2005), Maliar and Maliar (2013)).[2] There are two classes of methods that are particularly close to EFP. First, these are the methods for analyzing life-cycle models, developed in Krueger and Kubler (2004, 2006) and Hasanhodzic and Kotlikoff (2013). However, in a life-cycle economy, the terminal condition is either given or is a choice variable (as in a life-cycle model with bequests); see Ríos-Rull (1999) and Nishiyama and Smetters (2014) for reviews of the literature on life-cycle economies. In turn, terminal condition is unknown in our infinite horizon nonstationary economy and must be constructed in a way that ensures asymptotic convergence of the EFP approximation to the true infinite horizon solution. Second, EFP is related to a perturbation-based method of Schmitt-Grohé and Uribe (2012) that can solve models with anticipated shocks of a fixed horizon (e.g., shocks that happen each fourth or eight periods), however, unlike this method, EFP can handle anticipated shocks of any periodicity

[1] In turn, this literature on nonstationary models builds on mathematical tools developed for stationary models in Brock and Gale (1969), Brock (1971), Brock and Mirman (1972, 1973), McKenzie (1976), Mirman and Zilcha (1977), Brock and Majumdar (1978), Mitra and Zilcha (1981), among others.

[2] Surveys of numerical solution methods for stationary Markov models include Taylor and Uhlig (1990), Gaspar and Judd (1997), Judd (1998), Marimon and Scott (1999), Santos (1999), Miranda and Fackler (2002), Adda and Cooper (2003), Aruoba, Fernández-Villaverde and Rubio-Ramírez, (2006), Kendrik, Ruben-Mercado and Amman (2006), Canova (2007), Heer and Maußner (2010), Lim and McNelis (2008), Stachurski (2009), Kollmann, Maliar, Malin and Pichler (2011) and Maliar and Maliar (2014).

and duration.

Finally, EFP is related to numerical methods that construct a path for variables, in particular, shooting methods for deterministic economies introduced to economics in Lipton, Poterba, Sachs and Summers (1980) and an extended path method for economies with uncertainty proposed by Fair and Taylor (1983).[3] The related literature also includes solution methods for continuous time models studied in Chen (1999); a framework for characterizing equilibrium in life-cycle models with a deterministic aggregate path and idiosyncratic uncertainty, proposed by Conesa and Krueger (1999); a parametric path method of Judd (2002); and a predictive control method, developed in Grüne, Semmler and Stieler (2013); see also Atolia and Buffie (2009 a,b) for a careful discussion of shooting methods. The main shortcoming of this class of methods is that the assumption of certainty equivalence does not always provide sufficiently accurate approximations of expectation functions in nonlinear models. Adjemian and Juillard (2013) propose a stochastic extended path method that approximates expectation functions more accurately by constructing and averaging multiple paths for variables under different sequences of exogenous shocks. EFP differs from this literature in the way it deals with uncertainty, specifically, EFP constructs time-varying state-contingent decision functions that include stochastic shocks as additional arguments, whereas the above literature constructs one or several paths for endogenous variables.

The rest of the paper is as follows: In Section 2, we construct a class of nonstationary Markov models. In Section 3, we introduce EFP and provide its theoretical foundations. In Section 4, we describe the relation of EFP to the literature. In Section 5, we assess the performance of EFP in a nonstationary test model with a balanced growth path. In Section 6, we solve a collection of nonstationary applications; and finally in Section 7, we conclude.

2 A class of nonstationary Markov economies

We study a class of infinite-horizon nonlinear dynamic economic models in which preferences, technology and laws of motion for exogenous variables can change over time either deterministically or stochastically, according to a Markov process with possibly time-varying transition probabilities, or both. The constructed class of models is nonstationary because the optimal decision and value functions are time dependent. The existence theorems for stochastic growth models with time-varying fundamentals are established in Majumdar and Zilcha (1987), Mitra and Nyarko (1991) and Joshi (1997), among others.

2.1 The stochastic environment

Our exposition relies on standard measure theory notation; see, e.g., Stokey and Lucas with Prescott (1989), Santos (1999) and Stachurski (2009). Time is discrete and infinite, $t = 0, 1, ...$. Let $(\Omega, \mathcal{F}, P)$ be a probability space:

[3] Earlier literature was aware that methods solving for a path of variables can be used in the context of nonstationary problems. In particular, Lipton, Poterba, Sachs and Summers (1980, p.2) say "... we allow for a possibility that F [model's equations] may be time dependent (i.e., non-autonomous)". Also, Fair and Taylor (1983) explicitly assume that the model's equations are time dependent.

a) $\Omega = \Pi_{t=0}^{\infty}\Omega_t$ is a space of sequences $\varepsilon \equiv (\varepsilon_0, \varepsilon_1 ...)$ such that $\varepsilon_t \in \Omega_t$ for all t, where Ω_t is a compact metric space endowed with the Borel σ–field $\mathcal{E}_t$. Here, Ω_t is the set of all possible states of the environment at t and $\varepsilon_t \in \Omega_t$ is the state of the environment at t.

b) $\mathcal{F}$ is the σ–algebra on Ω generated by cylinder sets of the form $\Pi_{\tau=0}^{\infty}A_\tau$, where $A_\tau \in \mathcal{E}_\tau$ for all τ and $A_\tau = \Omega_\tau$ for all but finitely many τ.

c) P is the probability measure on $(\Omega, \mathcal{F})$.

We denote by $\{\mathcal{F}_t\}$ a filtration on Ω, where $\mathcal{F}_t$ is a sub σ–field of $\mathcal{F}$ induced by a partial history up of environment $h_t = (\varepsilon_0, ..., \varepsilon_t) \in \Pi_{\tau=0}^{t}\Omega_\tau$ up to period t, i.e., $\mathcal{F}_t$ is generated by cylinder sets of the form $\Pi_{\tau=0}^{t}A_\tau$, where $A_\tau \in \mathcal{E}_\tau$ for all $\tau \leq t$ and $A_\tau = \Omega_\tau$ for $\tau > t$. In particular, we have that $\mathcal{F}_0$ is the course σ–field $\{0, \Omega\}$, and that $\mathcal{F}_\infty = \mathcal{F}$. Furthermore, if Ω consists of either finite or countable states, ε is called a *discrete state process* or *chain*; otherwise, it is called a *continuous state process*. Our analysis focuses on continuous state processes, however, can be generalized to chains with minor modifications.

2.2 A nonstationary optimization problem

As an example, we consider a nonstationary stochastic growth model in which preferences, technology and laws of motion for exogenous variables change over time:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} E_0 \left[\sum_{t=0}^{\infty} \beta^t u_t (c_t) \right] \tag{1}$$

$$\text{s.t. } c_t + k_{t+1} = (1 - \delta) k_t + f_t (k_t, z_t), \tag{2}$$

$$z_{t+1} = \varphi_t (z_t, \varepsilon_{t+1}), \tag{3}$$

where $c_t \geq 0$ and $k_t \geq 0$ denote consumption and capital, respectively; initial condition (k_0, z_0) is given; $u_t : \mathbb{R}_+ \to \mathbb{R}$ and $f_t : \mathbb{R}_+^2 \to \mathbb{R}_+$ and $\varphi_t : \mathbb{R}^2 \to \mathbb{R}$ are possibly time-dependent utility function, production functions and law of motion for exogenous variable z_t, respectively; the sequence of u_t, f_t and φ_t for $t \geq 0$ is known to the agent in period $t = 0$; ε_{t+1} is i.i.d; $\beta \in (0, 1)$ is the discount factor; $\delta \in [0, 1]$ is the depreciation rate; and $E_t [\cdot]$ is an operator of expectation, conditional on a t-period information set.

Stationary models. A well-known special case of the general setup (1)–(3) is a stationary Markov models in which $u_t \equiv u$, $f_t \equiv f$ and $\varphi_t \equiv \varphi$. Such a model has a stationary Markov solution in which the value function $V (k_t, z_t)$ and decision functions $k_{t+1} = K (k_t, z_t)$ and $c_t = C (k_t, z_t)$ are both state-contingent and time-invariant; see, e.g., Stokey and Lucas with Prescott (1989, p. 391).

Nonstationary models. In a general case, a solution to the model (1)–(3) is nonstationary. The decision functions of endogenous variables c_t and k_t can be time-dependent for two reasons: first, because u_t and f_t change over time; and second, because φ_t and consequently, the transition probabilities of exogenous variable z_t change over time.

Remark 1. For presentational convenience, we assume that only z_t is a random variable following a Markov process with possibly time-varying transition probabilities, while the other model's parameters evolve in a deterministic manner, i.e., the sequence of u_t, f_t and φ_t for all $t \geq 0$ is deterministic. However, the quantitative framework we develop in the paper can be used to solve models in which β, δ, as well as the parameters of u_t, f_t and φ_t, are all random variables, following a Markov process with time-varying transition probabilities. In particular, in Section 6, we consider a version of the model in which δ follows a Markov process with time-varying transition probabilities.

2.3 Assumptions about exogenous variable

We provide some definitions that will be useful for explaining the assumption (3) about the Markov process for exogenous variable z_t; these definitions are standard and closely follow Stokey and Lucas with Prescott (1989, Ch. 8.2).

Definition 1. *(Stochastic process).* A stochastic process on $(\Omega, \mathcal{F}, P)$ is an increasing sequence of σ–algebras $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq ... \subseteq \mathcal{F}$; a measurable space $(Z, \mathcal{Z})$; and a sequence of functions $z_t : \Omega \to Z$ for $t \geq 0$ such that each z_t is $\mathcal{F}_t$ measurable.

Stationarity is commonly used assumption in economic literature.

Definition 2. *(Stationary process).* A stochastic process z on $(\Omega, \mathcal{F}, P)$ is called stationary if the unconditional probability measure, given by

$$P_{t+1,...,t+n}(C) = P(\{\varepsilon \in \Omega : [z_{t+1}(\varepsilon), ..., z_{t+n}(\varepsilon)] \in C\}), \tag{4}$$

is independent of t for all $C \in \mathcal{Z}^n$, $t \geq 0$ and $n \geq 1$.

A related notion is stationary (time-invariant) transition probabilities. Let us denote by $P_{t+1,...,t+n}(C|z_t = \overline{z}_t, ..., z_0 = \overline{z}_0)$ the probability of the event $\{\varepsilon \in \Omega : [z_{t+1}(\varepsilon), ..., z_{t+n}(\varepsilon)] \in C\}$, given that the event $\{\varepsilon \in \Omega : \overline{z}_t = z_t(\varepsilon), ..., \overline{z}_0 = z_0(\varepsilon)\}$ occurs.

Definition 3. *(Stationary transition probabilities).* A stochastic process z on $(\Omega, \mathcal{F}, P)$ is said to have stationary transition probabilities if the conditional probabilities

$$P_{t+1,...,t+n}(C|z_t = \overline{z}_t, ..., z_0 = \overline{z}_0) \tag{5}$$

are independent of t for all $C \in \mathcal{Z}^n$, $\varepsilon \in \Omega$, $t \geq 0$ and $n \geq 1$.

The assumption of stationary transition probabilities (5) implies the property of stationarity (4) provided that the corresponding unconditional probability measures exist. However, a process can be nonstationary even if transition probabilities are stationary, for example, a unit root process or explosive process is nonstationary; see Stokey and Lucas with Prescott (1989, Ch 8.2) for a related discussion. This kind of nonstationary processes is not studied explicitly in the present paper, i.e., we focus on nonstationarity that arises because transition probabilities change from one period to another.

In general, $P_{t+1,...,t+n}(C)$ and $P_{t+1,...,t+n}(C|\cdot)$ depend on the entire history of the events up to t (i.e., the stochastic process z_t is measurable with respect to the sub σ–field $\mathcal{F}_t$). However, history-dependent processes are difficult to analyze in a general case. It is of interest to distinguish special cases in which the dependence on history has relatively simple and tractable form. A well-known case is a class of Markov processes.

Definition 4. *(Markov process).* A stochastic process z on $(\Omega, \mathcal{F}, P)$ is (first-order) Markov if

$$P_{t+1,...,t+n}(C|z_t = \overline{z}_t, ..., z_0 = \overline{z}_0) = P_{t+1,...,t+n}(C|z_t = \overline{z}_t), \tag{6}$$

for all $C \in \mathcal{Z}^n$, $t \geq 0$ and $n \geq 1$.

The key property of a Markov process is that it is memoryless, namely, all past history $(z_t, ..., z_0)$ is irrelevant for determining the future realizations except of the most recent past z_t.

The literature typically assumed exogenous variables that are both stationary and Markov. As follows from (3), we maintain the assumption of Markov process (6), however, we relax the restriction of stationarity, namely, we allow for the case when transition probabilities (5) of z_t change over time. Below, we show an example that illustrates the type of stochastic processes that will be used in this paper for modeling exogenous variables.

Example 1. Consider a first-order autoregressive process

$$z_{t+1} = \rho_t z_t + \sigma_t \varepsilon_{t+1}, \tag{7}$$

where the sequences $(\rho_0, \rho_1, ...)$ and $(\sigma_0, \sigma_1, ...)$ are deterministically given at $t = 0$ and $\varepsilon_{t+1} \sim \mathcal{N}(0, 1)$. The conditional probability distribution $z_{t+1} \sim \mathcal{N}(\rho_t \overline{z}_t, \sigma_t)$ depends only on the most recent past $z_t = \overline{z}_t$ and is independent of history $(z_t, ..., z_0)$ as required by (6) and hence, the process is Markov. However, if ρ_t and σ_t change over time, then the distribution $\mathcal{N}(\rho_t \overline{z}_t, \sigma_t^2)$ depends not only on state $z_t = \overline{z}_t$ but also on a specific period t, so that transitions (5) are not stationary, and as a result, the process is nonstationary since it does not have time-invariant unconditional probability measure (4). If $\rho_t = \rho$ and $\sigma_t = \sigma$ for all t, then the conditional probability distribution $\mathcal{N}(\rho \overline{z}_t, \sigma^2)$ depends only on state $z_t = \overline{z}_t$ but not on time, and the transitions are stationary. If, in addition, $\rho < 1$, then the process is stationary in the sense (4). Finally, $\rho_t = 1$ for all t corresponds to a unit root process, which is nonstationary even if $\sigma_t = \sigma$ for all t; and $|\rho_t| > 1$ for all t leads to an explosive process. As we said earlier, unit root and explosive nonstationary processes are not explicitly studied in the present paper.

Remark 2. Mitra and Nyarko (1991) refer to a class of Markov processes with nonstationary transition probabilities as semi-Markov processes because of their certain similarity to Lévy's (1954) generalization of the Markov renewal process for the case of random arrival times; see Jansen and Manca (2006) for a review of applications of semi-Markov processes in statistics, operation research and other fields.

2.4 Assumptions about the utility and production functions

We make standard assumptions about the utility and production functions that ensure existence, uniqueness and interiority of a solution. Concerning the utility function u_t, we impose the following assumptions for each $t \geq 0$:

Assumption 1. *u_t is twice continuously differentiable on $\mathbb{R}_+$.*
Assumption 2. *$u_t' > 0$, i.e., u_t is strictly increasing on $\mathbb{R}_+$, where $u_t' \equiv \frac{\partial u_t}{\partial c}$.*
Assumption 3. *$u_t'' < 0$, i.e., u_t is strictly concave on $\mathbb{R}_+$, where $u_t'' \equiv \frac{\partial^2 u_t}{\partial c^2}$.*
Assumption 4. *u_t satisfies the Inada conditions $\lim\limits_{c \to 0} u_t'(c) = +\infty$ and $\lim\limits_{c \to \infty} u_t'(c) = 0$.*

Concerning the production function f_t, we make the following assumptions for each $t \geq 0$:

Assumption 5. *f_t is twice continuously differentiable on $\mathbb{R}_+^2$.*
Assumption 6. *$f_t'(k,z) > 0$ for all $k \in \mathbb{R}_+$ and $z \in \mathbb{R}_+$, where $f_t' \equiv \frac{\partial f_t}{\partial k}$.*
Assumption 7. *$f_t''(k,z) \leq 0$ for all $k \in \mathbb{R}_+$ and $z \in \mathbb{R}_+$, where $f_t'' \equiv \frac{\partial^2 f_t}{\partial k^2}$.*
Assumption 8. *f_t satisfies the Inada conditions $\lim\limits_{k \to 0} f_t'(k,z) = +\infty$ and $\lim\limits_{k \to \infty} f_t'(k,z) = 0$ for all $z \in \mathbb{R}_+$.*

Let us define a pure capital accumulation process $\{k_t^{\max}\}_{t=0}^{\infty}$ by assuming $c_t = 0$ for all t in (2) which for each history $h_t = (z_0, ..., z_t)$, leads to

$$k_{t+1}^{\max} = f_t\left(k_t^{\max}, z_t\right), \tag{8}$$

where $k_0^{\max} \equiv k_0$. We impose an additional joint boundedness restriction on preferences and technology by using the constructed process (8):

Assumption 9. *(Bounded objective function). $E_0\left[\sum_{t=0}^{\infty} \beta^t u_t\left(k_t^{\max}\right)\right] < \infty$.*

This assumption insures that the objective function (1) is bounded so that its maximum exists. In particular, Assumption 9 holds either (i) when u_t is bounded from above for all t, i.e., $u_t(c) < \infty$ for any $c \geq 0$ or (ii) when f_t is bounded from above for all t, i.e., $f_t(k, z_t) < \infty$ for any $k \geq 0$ and $z_t \in Z_t$. However, it also holds for economies with nonvanishing growth and unbounded utility and production functions as long as $u_t\left(k_t^{\max}\right)$ does not grow too fast so that the product $\beta^t u_t\left(k_t^{\max}\right)$ still declines at a sufficiently high rate and the objective function (1) converges to a finite limit.

2.5 Optimal program

Definition 1 *(Feasible program).* A feasible program for the economy (1)–(3) is a pair of adapted (i.e., F_t measurable for all t) processes $\{c_t, k_t\}_{t=0}^{\infty}$ such that given initial condition k_0 and history $h_\infty = (\varepsilon_0, \varepsilon_1 ...)$, they satisfy $c_t \geq 0$, $k_t \geq 0$ and (2) for all t.

We denote by $\Im^\infty$ a set of all feasible programs from given initial capital k_0 and given history $h_\infty = (\varepsilon_0, \varepsilon_1 ...)$. We next introduce the concept of solution of the studied model.

Definition 5 *(Optimal program).* A feasible program $\{c_t^\infty, k_t^\infty\}_{t=0}^\infty \in \Im^\infty$ is called optimal if

$$E_0 \left[\sum_{t=0}^{\infty} \beta^t \left\{ u_t \left(c_t^\infty \right) - u_t \left(c_t \right) \right\} \right] \geq 0 \tag{9}$$

for every feasible process $\{c_t, k_t\}_{t=0}^\infty \in \Im^\infty$.

Stochastic models with time-varying fundamentals are studied in Majumdar and Zilcha (1987), Mitra and Nyarko (1991) and Joshi (1997), among others. The existence results for this class of models have been established in the literature for a general measurable stochastic environment, i.e., in the absence of restriction of Markov process (3). In particular, this literature shows that, under assumptions Assumptions 1-9, there exists an optimal program $\{c_t^\infty, k_t^\infty\}_{t=0}^\infty \in \Im^\infty$ in the economy (1), (2), and it is both interior and unique; see Theorem 4.1 in Mitra and Nyarko (1991) and see Theorem 7 in Majumdar and Zilcha (1987). The results of this literature apply to us as well.

Remark 3. The existence of the optimal program in the economy (1)–(3) can be shown under weaker assumptions. For example, Mitra and Nyarko (1991) use a joint boundedness restriction on preferences and technology (the so-called *Condition E*) that is less restrictive than our Assumption 9; Joshi (1997) characterizes the optimal programs in nonconvex economies by relaxing our Assumptions 7 and 8, etc.

While the previous literature establishes the existence and uniqueness results for the constructed class of nonstationary model for a general non-Markov stochastic environment, it does not offer a practical approach for constructing time-dependent solutions in applications. In contrast, we will see that our additional Markov restriction (3) on z_t leads to a tractable class of nonstationary Markov models for which the solutions can be characterized both analytically and numerically.

3 Extended function path framework

We introduce a quantitative framework, which we call *extended function path* (EFP) framework, for approximating an optimal program in the nonstationary Markov economy (1)–(3). In Section 3.1, we present the EFP framework, and in Section 3.2, we develop its theoretical foundations.

3.1 Introducing extended function path framework

To approximate the optimal program in the nonstationary economy (1)–(3), we introduce a supplementary economy that becomes stationary in some remote period T.

Definition 5 *(T-period stationary economy).* A T-period stationary economy is the version of the economy (1)–(3) in which the utility and production functions and the laws of motions for exogenous variables are time invariant for $t \geq T$, i.e., $u_t = u$, $f_t = f$ and $\varphi_t = \varphi$ for all $t \geq T$.

The key idea of our EFP framework is to approximate an optimal program in the nonstationary

Markov economy (1)–(3) during the initial τ periods using a supplementary T-period stationary economy.

Algorithm 1: Extended function path.
Step 0. Initialization. Choose some $T \gg \tau$ and construct T-period stationary economy such that $u_t = u$, $f_t = f$ and $\varphi_t = \varphi$ for all $t \geq T$.
Step 1. Construct a stationary Markov solution , i.e., find a stationary capital function K satisfying: $u'(c) = \beta E\left[u'(c')(1 - \delta + f'(k', \varphi(z, \varepsilon')))\right]$ $c = (1 - \delta) k + f(k, z) - k'$ $c' = (1 - \delta) k' + f(k', \varphi(z, \varepsilon')) - k''$ $k' = K(k, z)$ and $k'' = K(k', \varphi(z, \varepsilon'))$.
Step 2. Construct a path for capital policy functions $(K_0, ..., K_T)$ that matches the terminal condition $K_T \equiv K$ and satisfies for $t = 0, ...T - 1$: $u_t'(c_t) = \beta E_t\left[u_{t+1}'(c_{t+1})(1 - \delta + f_{t+1}'(k_{t+1}, \varphi_t(z_t, \varepsilon_{t+1})))\right]$ $c_t = (1 - \delta) k_t + f_t(k_t, z_t) - k_{t+1}$ $c_{t+1} = (1 - \delta) k_{t+1} + f_{t+1}(k_{t+1}, \varphi_t(z_t, \varepsilon_{t+1})) - k_{t+2}$ $k_{t+1} = K_t(k_t, z_t)$ and $k_{t+2} = K_{t+1}(k_{t+1}, \varphi_t(z_t, \varepsilon_{t+1}))$.
The first τ functions $(K_0, ..., K_\tau)$ constitute an approximate solution and the remaining $T - \tau$ functions $(K_{\tau+1}, ..., K_T)$ are discarded.

A useful property of T-period stationary economy is that its optimal program is easy to characterize. First, since the economy (1)–(3) becomes stationary at T, the optimal program is stationary Markov for $t \geq T$, and Step 1 of EFP can be implemented by using conventional solution methods. Second, given the terminal condition produced by the T-period stationary economy, the sequence of $T - 1$ Euler equations identifies uniquely a path for decision functions for $t = 0, ..., T - 1$. To construct such a path, we can use backward induction, namely, given the capital function K_T, we use the Euler equation to compute the capital function K_{T-1} at $T - 1$; given K_{T-1}, we use it to compute K_{T-2}; and so on until the entire path $(K_T, ..., K_0)$ is constructed.

The term *extended path* indicates that EFP constructs a path of functions for larger time horizon T than the number of periods τ for which an approximate solution is actually needed, i.e., EFP extends the path from τ to T. In this respect, EFP is similar to extended path (EP) framework of Fair and Taylor (1983). By choosing sufficiently large T, both EFP and EP mitigate the effect of specific terminal condition on the approximation during the initial τ periods. In turn, the term *path* versus *function path* highlights the key difference between the EP and EFP methods: the former method constructs a path for variables, whereas the latter method constructs a path for decision functions. To approximate expectation functions, Fair and Taylor (1983) method relies on the assumption of certainty equivalence while the EFP method uses more accurate integration methods such as Monte Carlo, Gauss-Hermite quadrature and monomials methods. As a result, EP can accurately solve linear models with uncertainty (in such models, the certainty equivalence assumption leads to an exact approximation for integrals), as well as nonlinear models without uncertainty, whereas EFP can also accurately solve nonlinear models in which the certainty equivalence approach is either not applicable or leads to inaccurate solutions; in Section 4.3, we discuss the relation between EP and EFP in more details.

We implement EFP by using a combination of three techniques. First, to approximate decision functions, we use Smolyak (sparse) grids. Second, to approximate expectation functions, we use a nonproduct monomial integration rule. Finally, to solve for coefficients of the policy functions, we use a Gauss-Jacobi method, which is a derivative-free fixed-point-iteration method in line with Fair and Taylor (1983). The implementation details are described in Section 5.1 and Appendix B.

In Figure 1, we illustrate a sequence of functions (function path) produced by EFP for a version of the model (1)–(3) with exogenous growth due to labor augmenting technological progress (the model's parameterization and implementation details are described in Section 5).

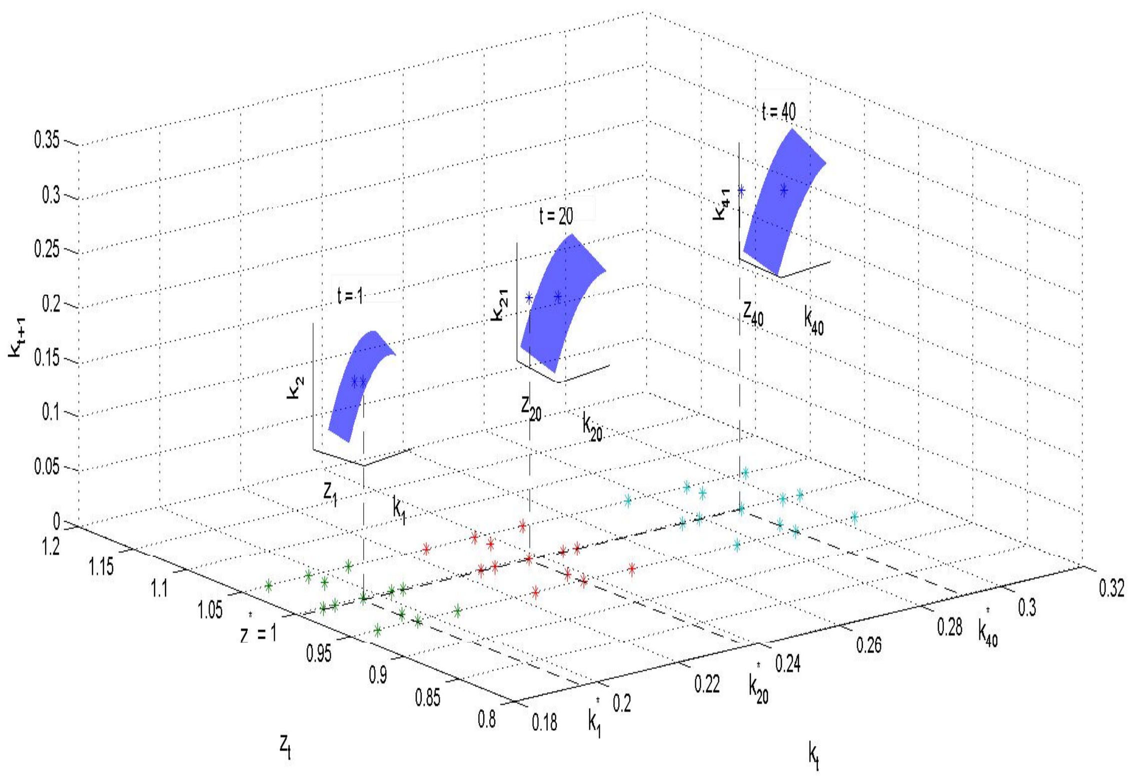

Figure 1. Function path, produced by EFP, for a growth model with technological progress

We plot the capital functions for periods 1, 20 and 40, (i.e., $k_2 = K_1(k_1, z_1)$, $k_{21} = K_{20}(k_{20}, z_{20})$ and $k_{41} = K_{40}(k_{40}, z_{40})$) which we approximate using Smolyak (sparse) grids. Here, in Step 1, we construct the capital function K_{40} by assuming that the economy becomes stationary in period $T = 40$; and in Step 2, we construct a path of the capital functions that $(K_1, ...K_{39})$ that matches the corresponding terminal function K_{40}. The Smolyak grids are shown by stars in the horizontal $k_t \times z_t$ plane. The domain for capital (on which Smolyak grids are constructed) and the range of the constructed capital function grow at the rate of labor augmenting technological progress.

Remark 4. In the paper, we analyze just one specific combination implementation of EFP but there are many ways in which EFP can be implemented: First, to construct decision functions, we can use a variety of grid techniques, integration rules, approximation methods, iteration schemes, etc. that are used by conventional solution methods. Second, to construct a function path, we can use any method that solves a system of nonlinear equations, including Newton-style solvers, Gauss-Siedel iteration used by shooting methods, Gauss-Jacobi iteration used by Fair and Taylor's (1983) method, etc. Since EFP constructs not just one but many decision functions (i.e., a separate decision function in each time period), we prefer techniques that have relatively low computational expense. Furthermore, to make EFP tractable in large-scale applications, we opt for techniques whose cost does not rapidly increase with the dimensionality of the problem (number of state variables). Maliar and Maliar (2014) survey techniques that are designed to deal with large-scale problems, including nonproduct sparse grids, simulated grids, cluster grids and epsilon-distinguishable-set grids; nonproduct monomial and simulation based integration methods, and derivative-free solvers.

Remark 5. The property of the T-period stationary economy that is essential for our analysis is that decision functions are stationary Markov at T. In our baseline implementation of EFP, we attain this property by assuming that the preferences, technology and laws of motion for exogenous variables do not change starting from $t = T$, i.e., $u_t = u_T$, $f_t = f_T$ and $\varphi_t = \varphi_T$ for all $t \geq T$. Instead, we can use other assumptions that lead to Markov decision functions at T, for example, we can assume that at T, the economy switches to a balanced growth path. Furthermore, we can assume that the economy arrives at a zero capital stock at T with the corresponding trivial Markov solution $k_t = 0$ for all $t \geq T$ (this case allows for standard interpretation of a finite horizon economy). Finally, we can use some T-period Markov terminal condition $K(k, z)$ without specifying explicitly an economic model that generates this terminal condition.

Remark 6. We have described a variant of EFP that constructs time-dependent capital functions $(K_0, ..., K_\tau)$. Similarly, we can formulate a variant of EFP that constructs time-dependent value functions $(V_0, ..., V_\tau)$. Such a value-iterative EFP first solves for $V_T = V_{T+1} = V$ for the T-period stationary economy and then it solves for a path $(V_{T-1}, ..., V_0)$ that satisfies the sequence of the Bellman equations for $t = 0, ..., T$ and that meets the terminal condition V_T of the T-period stationary economy.

3.2 Theoretical foundations of EFP framework

We now develop theoretical foundations of the EFP framework. We prove two theorems: Theorem 1 shows that the optimal program in the T-period stationary economy is given by a Markov process with possibly time-varying transition probabilities; and Theorem 2 shows that the optimal program of the T-period stationary economy converges to the optimal program of the original nonstationary Markov economy (1)–(3) as T increases.

Theorem 1 *(Optimal program of the T-period stationary economy). In the T-period stationary economy (1)–(3), the optimal program is given by a Markov process with possibly time-varying transition probabilities.*

Proof. Under Assumptions 1-9, first-order conditions (FOCs) are necessary for optimality. We will show that FOCs are also sufficient both to identify the optimal program and to establish its Markov structure. Our proof is constructive: it relies on backward induction and includes two steps that correspond to Steps 1 and 2 of EFP, respectively.

Step 1. At T, the economy becomes stationary and remains stationary forever, i.e., $u_t \equiv u$, $f_t \equiv f$ and $\varphi_t \equiv \varphi$ for all $t \geq T$. Thus, the model's equations and decision functions are time invariant for $t \geq T$. It is well known that under Assumptions 1-9, there is a unique stationary Markov capital function K that satisfies the optimality conditions that are listed in Step 1 of Algorithm 1; see, e.g., Stokey and Lucas with Prescott (1989, p. 391).

Step 2. Given the constructed T-period capital function $K_T \equiv K$, we define the capital functions $K_{T-1}, ..., K_0$ in previous periods by using backward induction. As a first step, we write the Euler equation for period $T-1$,

$$u'_{T-1}(c_{T-1}) = \beta E_{T-1}\left[u'_T(c_T)(1-\delta+f'_T(k_T, z_T))\right], \tag{10}$$

where c_{T-1} and c_T are related to k_T and k_{T+1} in periods T and $T-1$ by

$$c_{T-1} = (1-\delta)\,k_{T-1} + f_{T-1}(k_{T-1}, z_{T-1}) - k_T, \tag{11}$$

$$c_T = (1-\delta)\,k_T + f_T(k_T, z_T) - k_{T+1}. \tag{12}$$

By assumption (3), z_T follows a Markov process, i.e., $z_T = \varphi_T(z_{T-1}, \varepsilon)$. Furthermore, by construction of the decision function K in Step 1, we have that $k_{T+1} = K_T(k_T, z_T)$ is a Markov decision function. By substituting these two results into (10)–(12), we obtain a functional equation that defines k_T for each possible state (k_{T-1}, z_{T-1}). Therefore, the capital decisions at period $T-1$ are given by a state-contingent function $k_T = K_{T-1}(k_{T-1}, z_{T-1})$, i.e., capital decisions today are independent of history that leads to the current state. However, the constructed decision functions depend on the parameters of the utility and production functions and the law of motions for shocks in periods $T-1$ and T, and it is not generally true that $K_{T-1} \neq K_T$. By proceeding iteratively backward, we construct a sequence of state-contingent and possibly time-dependent capital functions $K_{T-1}(k_{T-1}, z_{T-1}), ..., K_0(k_0, z_0)$ that satisfies (10)–(12) for $t = 0, ..., T-1$ and that matches terminal function $K_T(k_T, z_T)$. Definitions 3 and 4 imply that k_{t+1} follows a Markov process with possibly time-varying transition probabilities. ■

We now show that the optimal program of the T-period stationary economy approximates arbitrary well the optimal program of the nonstationary economy (1)–(3) as T increases. Our analysis is related to the literature that shows asymptotic convergence of the optimal program of the finite horizon economy to that of the infinite horizon economy; see, e.g., McKenzie (1976) and Joshi (1997). This kind of convergence results is referred to as *turnpike theorems*. Our T-period stationary economy can be interpreted as a finite horizon economy characterized by a specific nonzero terminal condition; we therefore also refer to our convergence result as a turnpike theorem.

Let us fix history $h_\infty = (\varepsilon_0, \varepsilon_1 ...)$ and initial condition (k_0, z_0) and construct the productivity levels $\{z_t\}_{t=0}^{T}$ using (3). We then use the constructed sequence of capital functions

$K_0(k_0, z_0), ..., K_T(k_T, z_T)$ to generate the optimal program $\{c_t^T, k_t^T\}_{t=0}^{\infty}$ for the T-period stationary economy such that

$$k_{t+1}^T = K_t\left(k_t^T, z_t\right), \tag{13}$$

where $k_0^T = k_0$ and c_t^T satisfies the budget constraint (2) for all $t \geq 0$. Then, we have the following result.

Theorem 2 *(Turnpike theorem): For any real number $\varepsilon > 0$ and any natural number τ, there exists a threshold terminal date $T(\varepsilon, \tau)$ such that for any $T \geq T(\varepsilon, \tau)$,* we have

$$\left|k_t^{\infty} - k_t^T\right| < \varepsilon, \quad \text{for all } t \leq \tau, \tag{14}$$

where $\{c_t^{\infty}, k_t^{\infty}\}_{t=0}^{\infty} \in \Im^{\infty}$ is the optimal program in the nonstationary economy (1)–(3), and $\{c_t^T, k_t^T\}_{t=0}^{T}$ is the optimal program (13) in the T-period stationary economy.

Proof. See Appendix A. ■

The convergence is uniform: Our turnpike theorem states that for all $T \geq T(\varepsilon, \tau)$, the constructed nonstationary Markov approximation $\{k_t^T\}$ is guaranteed to be within a given ε-accuracy range from the true solution $\{k_t^{\infty}\}$ during the initial τ periods (for periods $t > \tau$, our approximation may become insufficiently accurate and exit the ε-accuracy range). The name *turnpike theorem* emphasizes the idea that turnpike is often the fastest route between two points which are far apart even if it is not a direct route. In terms of the studied model, this means that the optimal program of the T-period stationary economy $\{k_t^T\}$ follows for a long time the optimal program of the nonstationary economy $\{k_t^{\infty}\}$ (turnpike) and it deviates from turnpike only at the end to meet a given terminal condition (i.e. the final destination off turnpike).

In Figure 2, we illustrate the convergence of the optimal program of the T-period stationary economy to that of the original nonstationary economy. We again consider a version of the model with long-run growth due to labor augmenting technological progress (the parameterization of the model and the implementation details are described in Section 5). We fix the same initial condition and realization of shocks in all experiments. Here, k_t^{∞} denotes the true solution to the infinite-horizon nonstationary model (1)–(3) and k_L, k_T, k' and k'' denote the corresponding solutions to finite-horizon economies characterized by different terminal conditions.

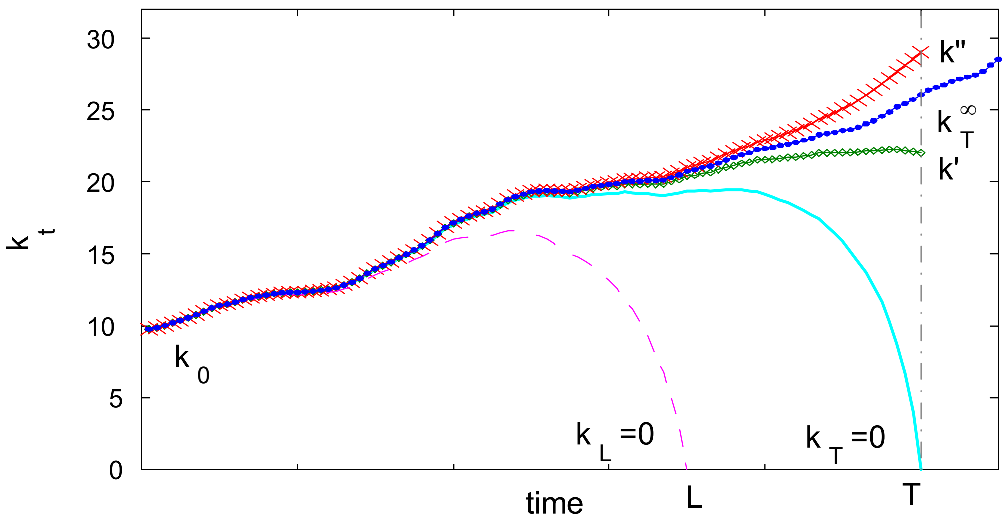

Figure 2. Convergence of the optimal program of T-period stationary economy

We observe the convergence of the simulated path of T-period stationary economy to that of the nonstationary economy under all terminal conditions considered. However, the convergence is faster under terminal conditions κ' and κ'', that are located relatively close to the true T-period capital $\{k_T^\infty\}$ of the nonstationary economy, then under zero terminal condition that is located farther away from the true solution. It is clear that a zero-capital terminal condition is not an efficient choice for constructing an approximation to the infinite horizon nonstationary economy, namely, in the infinite horizon economy, capital grows all the time, whereas in the finite horizon economy, capital needs to turn down at some point to meet a zero terminal condition. Our T-period stationary economy delivers more efficient terminal condition than the conventional finite-horizon approximation.

Remark 7. Our turnpike theorem shows the convergence of the optimal program of the T-period stationary economy to that of a nonstationary economy for a given initial condition and given history. It is classified as an early turnpike theorem in the literature; there are also medium and late turnpike theorems that prove the convergence by varying initial conditions and history, respectively; see McKenzie (1976) and Joshi (1997) for discussion. We do not prove other turnpike theorems for our T-period stationary economy because they are not directly related to the EFP quantitative framework introduced in the present paper.

4 Relation of EFP to the literature

EFP is related to three main steams of literature: (1) early theoretical literature that studies properties of solutions of non-Markov stochastic growth models; (2) literature on numerical methods for constructing solutions to stationary Markov models; (3) finally, literature on solving for a path for variables.

4.1 Early literature on stochastic growth models

The early literature provides theoretical foundations for stochastic growth models and characterizes the properties of their solutions; see Brock and Gale (1969), Brock (1971), Brock and Mirman (1972, 1973), Mirman and Zilcha (1977), Brock and Majumdar (1978), Mitra and Zilcha (1981), among others. In particular, Majumdar and Zilcha (1987), Mitra and Nyarko (1991), and Joshi (1997) study infinite horizon economies with deterministically time-varying utility and production functions similar to ours. However, this literature is limited to purely theoretical analysis and does not offer practical methods for constructing their nonstationary solutions in applications.

Our main contributions relative to that literature are that we distinguish a tractable Markov class of nonstationary models and propose an EFP framework for analyzing quantitative implications of such models. In addition, we show new formal results. First, our Theorem 1 establishes Markov structure of the optimal program in the T-period stationary economy while the previous literature establishes similar results for a finite-horizon economy with a zero terminal condition; e.g., Mitra and Nyarko (1991, Theorem 4.3). Second, our Theorem 2 (turnpike theorem) focuses on terminal condition that is generated by a Markov solution to a class of T-period stationary economies, whereas the turnpike theorems existing in the literature assumes a specific zero terminal condition, $k_T = 0$ representing a finite-horizon economy; see, e.g., Majumdar and Zilcha (1987), Mitra and Nyarko (1991) and Joshi (1997). Our two theorems provide theoretical foundations for the EFP framework.

4.2 Methods constructing Markov decision functions

The mainstream of economic literature relies on stationary Markov models. There is a variety of methods for constructing solutions to such models, in particular, projection methods (see, e.g., Judd (1992), Christiano and Fisher (2000), Maliar and Maliar (2015)); perturbation methods (see, e.g., Judd and Guu (1993), Collard and Juillard (2001), Schmitt-Grohé and Uribe (2004)); and stochastic simulation methods (see, e.g., Den Haan and Marcet (1990), Judd, Maliar and Maliar (2011)); and numerical dynamic programming methods, in particular, those designed to deal with large scale applications (see, e.g., Smith (1993), Rust (1996), Carroll (2005), Maliar and Maliar (2013)). For surveys of such methods, see Taylor and Uhlig (1990), Rust (1996), Gaspar and Judd (1997), Judd (1998), Marimon and Scott (1999), Santos (1999), Miranda and Fackler (2002), Adda and Cooper (2003), Aruoba, Fernández-Villaverde and Rubio-Ramírez (2006), Kendrik, Ruben-Mercado and Amman (2006), Canova (2007), Heer and Maußner (2010), Lim and McNelis (2008), and Stachurski (2009), Den Haan (2010). In particular, Kollmann, Maliar, Malin and Pichler (2011) and Maliar and Maliar (2014) survey numerical methods for analyzing problems with high dimensionality. The conventional methods for constructing stationary Markov solutions cannot generally be used for solving models with time varying parameters studied in the present paper, however, the techniques used by these conventional methods can be used as ingredients of EFP, including a variety of grid techniques, integration methods, numerical solvers, etc.

There are three groups of Markov methods that EFP is particularly close to. First of all, EFP is related to numerical methods that construct decision functions in life-cycle models as in Krueger and Kubler (2004, 2006) and Hasanhodzic and Kotlikoff (2013). The decision functions in such models change from one generation to another, and the sequence of the generation-specific decision functions resembles a function path constructed by EFP; see Ríos-Rull (1999) and Nishiyama and Smetters (2014) for reviews of the literature on life-cycle economies. The difference is that terminal condition is either known in the life-cycle economy or is a choice variable (as in an economy with bequests), while it is unknown in our infinite-horizon economy and must be constructed in the way that ensures the convergence of an EFP approximation to the true nonstationary solution.

Furthermore, EFP is related to economic literature that studies Markov nonstationary models with balanced growth paths; see King, Plosser and Rebello (1988) for restrictions on preferences and technology that are consistent with a balanced growth path. However, this class of models is limited; for example, models with labor augmenting technological progress are generally consistent with a balanced growth path but not models with either capital augmenting or Hicks neutral technological progress. There are examples of constructing balanced growth path for some models that do not satisfy the restrictions of King, Plosser and Rebello (1988) but they are also limited.[4]

Finally, EFP is related to the literature that incorporates certain kinds of nonstationarity by augmenting the economic models to include additional state variables. In particular, Bloom (2009), Fernández-Villaverde and Rubio-Ramírez (2010), Fernández-Villaverde, Guerrón-Quintana and Rubio-Ramírez (2010), among others, argue that the behavior of real-world economies is affected by degrees of uncertainty and introduce models with stochastic volatility. Furthermore, Davig and Leeper (2009), Farmer, Waggoner and Zha (2011), Foerster, Rubio-Ramírez, Waggoner and Zha (2013) and Zhong (2015), among others, advocate periodic unanticipated changes in regimes. In particular, a recent paper of Schmitt-Grohé and Uribe (2012) introduces a quantitative framework that allows for anticipated exogenous shocks of a fixed periodicity and length. The key difference of our analysis from this literature in that it allows for time dependence of the model itself while the above literature expands the state space of time-invariant models.

4.3 Methods constructing a path for variables

The EFP framework is related to numerical methods that construct a path for variables in deterministic economies. To illustrate such methods, let us abstract from uncertainty by assuming that f_t depends on k_t but not on z_t. By substituting c_t and c_{t+1} from (2) into the Euler equation of (1)–(3), we obtain a second-order difference equation,

$$\begin{aligned} u_t'((1-\delta)\,k_t + f_t\,(k_t) - k_{t+1}) \\ = \beta\left[u_{t+1}'((1-\delta)\,k_{t+1} + f_{t+1}\,(k_{t+1}) - k_{t+2})(1-\delta + f_{t+1}'\,(k_{t+1}))\right]. \quad (15) \end{aligned}$$

Initial condition k_0 is given. Let us choose a sufficiently large T and fix some k_{T+1} (typically, the literature assumes that the economy arrives in the steady state $k_{T+1} = k^*$). This yields

[4] Two examples are as follows: Maliar and Maliar (2004) shows the existence of a balanced growth path in a model with endogenous growth and cycles by removing a common stochastic trend representing randomly arriving technological innovations; and Maliar and Maliar (2010) constructs a balanced growth path in a model with capital-skill complementarity and several types of technical progress by imposing additional restrictions on growth rates of variables.

a system of T nonlinear equations (15) with T unknowns $\{k_1, ..., k_T\}$. The turnpike theorem implies that in initial τ periods, the solution to this system is insensitive to a specific terminal condition used if $\tau \ll T$.

It is possible to solve the system (15) numerically by using a Newton-style or other numerical solvers, however, it could be expensive. As an alternative, the literature developed numerical methods that exploit the recursive structure of the system (15). A well known is a class of shooting methods that solve for path $(k_1, ..., k_T)$by using Gauss-Siedel iteration. There are two type of shooting methods: a forward shooting and a backward shooting. A typical forward shooting method expresses k_{t+2} in terms of k_t and k_{t+1} and constructs a forward path $(k_1, ..., k_{T+1})$; it iterates on k_1 until the path reaches a given terminal condition $k_{T+1} = k^*$. In turn, a typical reverse shooting method expresses k_t in terms of k_{t+1} and k_{t+2} and constructs a backward path $\{k_T, ..., k_0\}$; it iterates on k_T until the path reaches a given initial condition k_0. Shooting methods are introduced to economics in Lipton, Poterba, Sachs and Summers (1980) who also noticed their potential for solving nonstationary models. A shortcoming of shooting methods is that they tend to produce explosive paths, in particular, forward shooting methods; see Atolia and Buffie (2009 a, b) for a careful discussion and possible treatments of this problem.

Fair and Taylor (1983) introduced an extended path method that can be used to solve economic models with uncertainty. Their method relies on a certainty-equivalence approximation, namely, it replaces expectation of a function across states with a value of the function in the expected state. In terms of the economy (1)–(3), this means

$$E_t \left[u'_{t+1} \left(c_{t+1}\right) \left(1 - \delta + f'(k_{t+1}, z_{t+1})\right)\right] \approx u'_{t+1} \left(c_{t+1}\right) \left(1 - \delta + f'_{t+1}(k_{t+1}, E_t \left[z_{t+1}\right])\right). \quad (16)$$

This kind of approximation is exact for linear and linearized models, and it can be sufficiently accurate for models that are close to linear; see Cagnon and Taylor (1990), and Love (2010). However, it becomes highly inaccurate when either volatility and/or the degrees of nonlinearity increase; see our accuracy evaluations in Section 5.

To avoid explosive behavior, Fair and Taylor's (1983) method iterates on the economy's path at once in line with Gauss-Jacobi iteration. Namely, it guesses the economy's path $(k_1, ..., k_{T+1})$, substitute the quantities for $t = 1, ...T + 1$ it in the right side of T Euler equations (15), respectively, and obtains a new path $(k_0, ..., k_T)$ in the left side of (15); and it iterates on the path until the convergence is achieved. Also, Fair and Taylor (1983) propose a simple procedure for determining T that is sufficient to insure that a specific terminal condition used does not affect the quality of approximation, namely, they suggested to increase T (i.e., extend the path) until the solution in the initial period(s) becomes insensitive to further increases in T. In Appendix C, we describe a specific implementation of Fair and Taylor's (1983) method, which we used in Section 5 for comparison with EFP.

There are other methods in the literature that solve for path. In particular, Chen (1999) propose methods for finding a solution to continuous time models. Conesa and Krueger (1999) introduce a framework for characterizing equilibria in nonstationary life-cycle models in which the aggregate economy's path is deterministic but there is idiosyncratic uncertainty. Judd (2002) proposes a parametric path method that approximates a deterministic path using a family of polynomial functions. Heer and Maußner (2010) implement Fair and Taylor's (1983) method using a Newton-style solver. Finally, Grüne, Semmler and Stieler (2013) develop a nonlinear model predictive control method that solves for a path of variables by maximizing the objective function with a numerical solver directly, without using first-order conditions. Applications of path methods in economics are numerous, e.g., Chen, Imrohoroğlu and Imrohoroğlu (2006), Bodenstein, Erceg and Guerrieri (2009), Coibion, Gorodnichenko and Wieland (2011), Braun and Körber (2011) and Hansen and Imrohoroğlu (2013).

Adjemian and Juillard (2013) propose a modification of Fair and Taylor's (1983) method, called stochastic extended path method, that improves on accuracy of approximation of conditional expectation functions. The main idea of their method is to construct a tree of all possible future shocks and to solve for multiple paths for variables on all branches of the tree. The expectation functions is approximated with a weighted average of the corresponding variables on multiple paths. The number of tree branches and paths grows exponentially with the path length and so does the cost of this method but the authors propose a clever way of reducing the cost by restricting attention to paths that have highest probability of occurrence.

The EFP construction of function path is similar to the construction of variables path in the above literature however EFP differs from this literature in the object it constructs and in the way it deals with uncertainty. Namely, EFP constructs a sequence of Markov state-contingent decision functions that include stochastic shocks as one of the arguments rather than solving for a path under some sequences of shocks. In this respect, EFP is similar to conventional numerical approaches that construct state-contingent solutions to stationary Markov models.

5 Assessing EFP accuracy in a test model with balanced growth

To assess the quality of approximations produced by EFP in nonstationary environments, we need a test application for which a sufficiently accurate solution is available. We use a version of the model (1)–(3) with labor augmenting technological progress parameterized by Cobb-Douglas utility and production functions,

$$u_t(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \text{and} \quad f_t(k,z) = zk^{\alpha}A_t^{1-\alpha}, \tag{17}$$

where $\gamma > 0$ and $\alpha \in (0,1)$; $A_t = A_0 g_A^t$ represents a labor augmenting technological progress with an exogenous constant growth rate $g_A \geq 1$. The process for the productivity level (3) is given by

$$\ln z_{t+1} = \rho \ln z_t + \sigma \varepsilon_{t+1}, \qquad \varepsilon_{t+1} \sim \mathcal{N}(0,1), \tag{18}$$

where $\rho \in (-1,1)$, $\sigma \in (0,\infty)$. This version of the model is consistent with balanced growth and can be converted into a stationary model; see King, Plosser and Rebelo (1988). We can first solve the stationary model very accurately using conventional solution methods, and we can then recover an accurate solution to the original nonstationary model (1)–(3) to be used for a comparison; see Appendix 7 for a description of the stationary model.

5.1 Implementation details of EFP

EFP solves the original, nonstationary growth model (1)–(3) without converting it into stationary. The path of function produced by EFP is shown in Figure 1. Below, we discuss some implementation details of EFP; further implementation details are provided in Appendix B.

First, EFP begins by constructing a sequence of grids for $t = 0, ..., T$ on which a sequence (path) of the decision functions will be approximated. An important practical question is where the grids must be centered in $t = 0, ..., T$. In the conventional stationary model, we typically center a grid in the deterministic steady state. However, in a growing economy, the steady state does not exist. To address this issue, we define an analogue of steady state for non-stationary economies as a path for the model's variables that constitutes a solution to an otherwise identical deterministic model in which all shocks are shut down. We call such a solution a *growth path* and we denote it by "$*$" superscript. For example, in Figure 1, we show growth path for capital k_1^*, k_{20}^* and k_{40}^* for periods 1, 20 and 40, respectively; see the centers of Smolyak grids in the $k_t \times z_t$ plane.

In the special case of balanced growth model (1)–(3), the growth path can be constructed analytically, by using the deterministic steady state of the corresponding stationary model. Namely, in the stationary model, the steady state is given by $k_0^* \equiv A_0 \left(\dfrac{g_A^\gamma - \beta + \delta\beta}{\alpha\beta} \right)^{1/(\alpha-1)}$, and in $t = 1, ..., T$, it evolves as $k_t^* = k_0^* g_A^t$. In unbalanced growth models, the growth path must be in general constructed using numerical techniques and requires to specify initial and terminal conditions; see Section 6.1 for a discussion and examples.

Second, EFP requires us to specify a terminal condition in the form of T-period decision functions. (For example, in Figure 1, the terminal period is $T = 40$ and the terminal decision function is K_{40}). What terminal condition do we choose? Again, for a special case of balanced growth model, it is possible to infer the "exact" terminal condition from the solution to the stationary model; see Appendix D for details. However, in a general case, a balanced growth path and an appropriate terminal condition is unknown. To assess the role of the terminal condition in the accuracy of solutions, we compare two different EFP solutions: in one solution, we use the "exact" terminal condition, which is the T-period decision functions inferred from the balanced growth model; and in the other solution, we use a stationary Markov solution to a T-period stationary economy which stops growing at T.

Finally, our turnpike analysis states that we can always find a sufficiently large T so that the approximation produced by EFP is sufficiently accurate during the first τ periods. But how do we choose T and τ in applications? A popular implementation of Fair and Taylor's (1983) method builds on $\tau = 1$, namely, we first construct a path between given k_0 and k_{T+1} and we take only k_1 from the constructed path; we then constructs a path between k_1 and k_{T+2} and takes only k_2; and so on until the path of a required length is constructed. In contrast, we implement EFP by using much larger values of τ such as 50 or 100 by considering also larger T's in order to economize on cost.

5.2 A comparison of four solution methods

We solve the nonstationary growth model (1)–(3) using four alternative solution methods: (1) a conventional method that constructs a solution to the stationary model with a balanced growth path; (2) EFP method that solves a nonstationary model directly; (3) Fair and Taylor's (1983) method that uses a certainty equivalence assumption (16) to approximates expectation functions; (4) a naive method that replaces a nonstationary model with a sequence of stationary models and that solves such models one by one. The naive method differs from EFP in that it neglects the connection between the decision functions of different periods. We refer to the solutions produced by the four methods considered as *exact solution*, *EFP solution*, *Fair and Taylor's solution* and *naive solution*, respectively.

The solution that we call *exact* is not exact but very accurate, namely, unit-free maximum residuals in the model's equations are of order 10^{-6} on a stochastic simulation of 10,000 observations; see Maliar and Maliar (2014) and Judd, Maliar, Maliar and Valero (2014) for accuracy evaluations of Smolyak methods. It will suffice for us to show that EFP can attain the same accuracy levels for nonstationary models as the state-of-the-art conventional solution methods do for similar stationary models.

For all experiments, we fix $\alpha = 0.36$, $\beta = 0.99$, $\delta = 0.025$ and $\rho = 0.95$. The remaining parameters are set in the benchmark case at $\gamma = 5$, $\sigma_\varepsilon = 0.03$, $g_A = 1.01$ and $T = 200$, and we vary these parameters across experiments. For all simulations, we use the same initial condition and the same sequence of productivity shocks.

Our code is written in MATLAB 2013a, and we use a desktop computer with Intel(R) Core(TM) i7-2600 CPU (3.40 GHz) with RAM 12GB. The running times for EFP can be reduced considerably if we use parallelization (our iteration, which is in line with Gauss-Jacobi method, is naturally parallelizable).

5.2.1 Critical role of expectations in the accuracy of solutions

In the left panel of Figure 3, we represent the growing time-series solutions for the four solution methods, as well as the (steady state) growth path for capital. In the right panel, we plot the time series solutions after detrending the growth path.

As is evident from the both panels, the EFP solution and the exact solution are visibly indistinguishable except at the end of the time horizon – the last $10 - 15$ periods, which suggests that the accuracy range τ may be large in this particular example. Fair and Taylor's (1983) and naive methods produce the solutions that are visibly below the exact solution; and the naive solution is the least accurate of all.

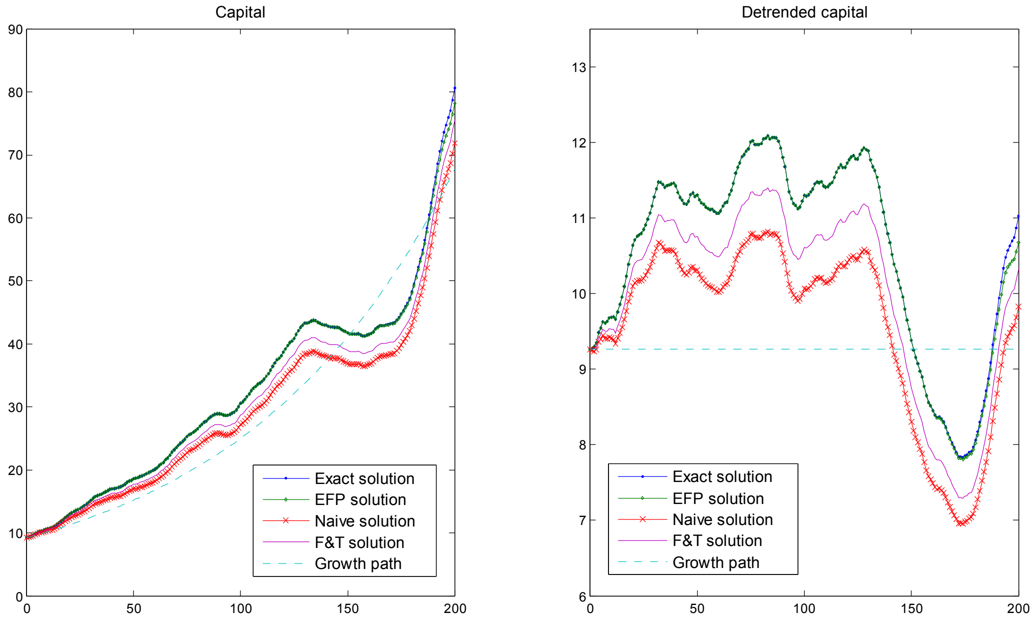

Figure 3. Comparison of the solution methods for the test model with balanced growth

We assess the accuracy of the constructed solutions numerically. We first simulate each of the four solutions for 100 times and we then compute the mean and maximum absolute differences in $\log 10$ units between the exact solution and the remaining three solutions across 100 simulations for the intervals $[0, 50]$, $[0, 100]$, $[0, 150]$, $[0, 175]$, and $[0, 200]$. This kind of accuracy evaluation shows how the accuracy of the approximations depends on τ. We report the accuracy results in Table 1, where we also report the time needed for computing and simulating 100 solutions of length T (in seconds).

In Table 1, the difference between the exact and EFP solutions is less than $10^{-6} \approx 0.0001\%$ over the first 50 periods for the three experiments considered (differing in time horizon and terminal condition). Thus, the EFP method delivers a remarkably accurate solution for $\tau = 50$ with time horizon $T = 200$.

Furthermore, the differences between the exact solution and Fair and Taylor's (1983) solution are around $10^{-1.6} \approx 2.5\%$ in Table 1. Fair and Taylor's (1983) method has relatively low accuracy because formula (16) used for approximating conditional expectation is inaccurate. Fair and Taylor's (1983) method is more accurate for models with a smaller variance of shocks and /or smaller degrees of nonlinearities. For example, we assess the difference between the exact solution and Fair and Taylor's (1983) solutions for the model with $\gamma = 1$, $\sigma_\varepsilon = 0.01$, $g_A = 1.01$ and $T = 200$, and we found that such a difference is around 0.1% (this experiment is not reported).

Finally, the difference between the exact and naive solutions in Table 1 can be as large as 10%. The poor performance of the naive may seem surprising given that such a method does take into account the technology growth when constructing solutions. Namely, a naive method solves each t-period stationary model by assuming that productivities at t and $t+1$ are correctly given by $A_t = A_0 g_A^t$ and $A_{t+1} = A_0 g_A^{t+1}$, respectively. Why is the naive method so

Table 1: Comparison of four solution methods.

	Fair-Taylor (1983) method, $\tau = 1$		Naive method	EFP method $\tau = 1$			EFP method $\tau = 200$		
Terminal condition	Steady state	Steady state	-	Balanced growth	T-period stationary		Balanced growth	T-period stationary	
T	200	400	200	200	200	400	200	200	400
Mean errors across t periods in $\log_{10}$ units									
$t \in [0, 50]$	-1.60	-1.60	-1.36	-7.30	-6.97	-7.15	-7.23	-6.75	-7.01
$t \in [0, 100]$	-1.42	-1.42	-1.19	-7.06	-6.81	-6.98	-7.03	-6.19	-6.81
$t \in [0, 150]$	-1.34	-1.35	-1.11	-6.96	-6.73	-6.91	-6.94	-5.47	-6.73
$t \in [0, 175]$	-1.32	-1.32	-1.09	-6.93	-6.71	-6.89	-6.91	-5.09	-6.70
$t \in [0, 200]$	-1.30	-1.31	-1.07	-6.91	-6.69	-6.87	-6.90	-4.70	-6.68
Maximum errors across t periods in $\log_{10}$ units									
$t \in [0, 50]$	-1.29	-1.29	-1.04	-6.83	-6.63	-6.81	-6.82	-6.01	-6.42
$t \in [0, 100]$	-1.18	-1.18	-0.92	-6.69	-6.42	-6.68	-6.68	-4.39	-5.99
$t \in [0, 150]$	-1.14	-1.14	-0.89	-6.66	-6.39	-6.67	-6.66	-2.89	-5.98
$t \in [0, 175]$	-1.14	-1.13	-0.89	-6.66	-6.40	-6.66	-6.66	-2.10	-5.98
$t \in [0, 200]$	-1.14	-1.13	-0.89	-6.66	-6.37	-6.66	-6.66	-1.45	-5.92
Running time, in seconds									
Solution	1.2(+4)	6.1(+4)	28.9	216.5	8.6(+3)	1.9(+4)	104.9	99.1	225.9
Simulation	-	-	2.6	2.6	2.6	5.8	2.6	2.8	5.7
Total	1.2(+4)	6.1(+4)	31.5	219.2	8.6(+3)	1.9(+4)	107.6	101.9	231.6

Notes: "Mean errors" and "Marimum errors" are, respectively, mean and maximum unit-free absolute difference between the exact solution for capital and the solution delivered by a method in the column. The difference between the solutions is computed across 100 simulations.

inaccurate? The reason is that in each time period, the naive method computes a stationary solution under the assumption that technology will remain at the current levels A_t and A_{t+1} forever, meanwhile in the true nonstationary economy, technical change continues forever. As a result, in the former case, the agent is "unaware" about the future permanent productivity growth and has expectations that are systematically more pessimistic than those of the agent in the true nonstationary growing economy. It was pointed out by Cooley, Leroy and Raymon (1984) that naive-style solution methods are logically inconsistent: agents are unaware about a possibility of parameter changes when they solve their optimization problems, however, they are later confronted with parameter changes in simulations. Our analysis suggests that this effect is particularly large in growing economies. We conclude that approximating expectation functions accurately is critical for constructing accurate solutions to nonstationary growth models.

5.2.2 Terminal condition and the "tail" of simulation

As Figure 3 shows, the exact and EFP solutions differ in the tail considerably; this difference is especially well seen for the detrended time series in the right panel. The difference in the tail is due to the difference in the terminal conditions. Namely, to construct the exact solution, we assume that the economy grows forever while to construct the EFP solution, we assume that it stops growing at T. If we use the same terminal conditions in both cases, then the EFP solution would be visually indistinguishable from the exact solution everywhere in the figure.

In Table 1, we first consider a version of EFP that constructs the function path under $\tau = 1$ (this is similar to the implementation of Fair and Taylor's (1983) method used in the literature). Namely, given the capital function K_T in the T-period stationary economy, we solve for decision functions for $t = 0, ..., T-1$, store K_0 and discard the rest of the functions. Next, given K_{T+1}, we solve for decision functions for $t = 1, ..., T$, store K_1 and discard the rest of the functions. We proceed forward until the whole path $(K_0, ...K_T)$ is constructed. We consider two different lengths of time horizon $T = 200$ and $T = 400$, and we consider two different terminal conditions: one comes from the solution of the stationary balanced growth model (and can be viewed as "exact" terminal condition) and the other comes from the T-period stationary economy (and is far from the "exact" terminal condition).

EFP method with $\tau = 1$ is very accurate in the studied example independently of specific terminal condition used, namely, the EFP solution differs from the exact solution by less than $10^{-6} = 0.0001\%$. This result illustrates that the effect of specific terminal condition on the very first element of the path $\tau = 1$ is negligible if the length of the path T is sufficiently large.

5.2.3 How to extend the path

A shortcoming of the described version of EFP with $\tau = 1$ is its high computational expense: the running time under $T = 200$ and $T = 400$ is 15 and 30 minutes, respectively. The cost is high because we need to recompute a sequence of decision functions each time when we extend the path by one period ahead. Effectively, we solve the model T times and not just once.

Our turnpike theorem suggests a cheaper version of EFP in which we construct a longer function path but do it just once; the results for such EFP method are provided in the last three columns of Table 1. We now observe that the terminal condition plays a critical role in the accuracy of solutions near the tail. If we use the terminal condition from the T-period stationary economy, the errors increase as we advance in time and reach nearly 4% at the end of simulation. In contrast, if we use the terminal condition from the stationary balanced growth model, the EFP solution is very accurate everywhere including the tail. Finally, the most important result is shown in the last column. If we construct a function path of length $T = 400$, however, use only the first $\tau = 200$ decision functions, the solution for the first $\tau = 200$ periods is almost as accurate as that produced by a sequence of functions with $\tau = 1$. This is true even though we use the terminal condition from the T-period stationary economy that is far away from the exact solution. We draw attention to the fact that constructing a longer path is relatively inexpensive: the running time increases from about 2 minutes to 4 minutes under $T = 200$ and $T = 400$, respectively.

5.2.4 Cost of finding solution and cost of simulation

An important advantage of EFP relatively to methods that solve for a path of variables is its low simulation cost. Under EFP, we construct a path for decision functions just once, and we can use the constructed functions to simulate the model as many times as needed under different sequences of shocks. In contrast, under Fair and Taylor's (1983) and other methods that solve for a path of variables, the solution and simulation steps are combined: in order to produce a new simulation, we need to entirely recompute a solution to the model under a different sequence of shocks. The time that EFP needs to compute a solution and simulate it 100 times is about 2 and 4 minutes for $T = 200$ and $T = 400$, respectively, while the respective times for Fair and Taylor's (1983) method are 20 and 60 minutes.

5.2.5 Sensitivity analysis

On the basis of the results in Table 1, we advocate the version of EFP that constructs a sufficiently long path just once by using $T \gg \tau$. In Table 2, we assess the accuracy of this preferred EFP version with $T = 400$ under different parameterizations. As a terminal guess, we use decision rules produced by the T-period stationary economy. We consider several combinations of the values of the parameters $\{\gamma, \sigma_\varepsilon, g_A\}$ such that $\gamma \in \{0.1; 1; 5; 10\}$, $\sigma_\varepsilon \in \{0.01; 0.03\}$ and $g_A \in \{1; 1.01; 1.05\}$. Our benchmark values (see "Model 1") are $\{\gamma; \sigma_\varepsilon; g_A\} = \{5; 0.03; 1.01\}$.

Depending on a specific parameterization, the difference between the exact and EFP solutions in the model's equations for $\tau = 200$, vary between $10^{-6} = 0.0001\%$ and $10^{-7} = 0.00001\%$, which are high accuracy levels. The running time for all cases except for Model 5 is between 155 seconds and 306 seconds, which is reasonable. There is one case when the computational time increases to 842 seconds, which corresponds to to a low degree of risk aversion parameter $\gamma = 0.1$. (We find that with low degree of risk aversion, the convergence is more fragile and we had to decrease the damping parameter from $\xi = 0.05$ to $\xi = 0.1$). Overall, our results suggests that the EFP method can solve nonstationary growth models both accurately and reliably in a wide range of the model's parameters at a relatively modest cost.

Table 2: Sensitivity analysis for the EFP method.

Parameters	Model 1	Model 2	Model 3	Model 4	Model 5	Models 6	Model 7
γ	5	5	5	5	0.1	1	10
σ_ϵ	0.03	0.03	0.03	0.01	0.01	0.01	0.01
g_A	1.01	1.00	1.05	1.01	1.01	1.01	1.01
Mean errors across t periods in $\log_{10}$ units							
$t \in [0, 50]$	-7.01	-6.67	-7.34	-7.03	-7.03	-6.61	-7.30
$t \in [0, 100]$	-6.82	-6.44	-7.25	-6.84	6.92	-6.48	-7.08
$t \in [0, 150]$	-6.73	-6.33	-7.22	-6.76	-6.89	-6.43	-6.98
$t \in [0, 175]$	-6.70	-6.29	-7.22	-6.74	-6.87	-6.41	-6.95
$t \in [0, 200]$	-6.68	-6.26	-7.21	-6.72	-6.87	-6.37	-6.93
Maximum errors across t periods in $\log_{10}$ units							
$t \in [0, 50]$	-6.42	-6.31	-7.13	-6.66	-6.08	-6.24	-6.81
$t \in [0, 100]$	-5.99	-6.12	-7.05	-6.54	-5.97	-6.18	-6.36
$t \in [0, 150]$	-5.98	-6.04	-7.05	-6.52	-5.97	-6.18	-6.35
$t \in [0, 175]$	-5.98	-6.01	-7.05	-6.52	-5.97	-6.13	-6.33
$t \in [0, 200]$	-5.92	-5.99	-7.05	-6.51	-5.96	-5.88	-6.24
Running time, in seconds							
Solution	225.9	150.0	193.0	216.98	836.5	300.7	245.9
Simulation	5.6	5.7	5.8	5.66	5.6	5.6	5.7
Total	231.6	155.7	198.8	222.64	842.1	306.3	251.6

Notes: "Mean errors" and "Marimum errors" are, respectively, mean and maximum unit-free absolute difference between the exact solution for capital and the solution delivered by EFP under the parameterization in the column. The difference between the solutions is computed across 100 simulations. The time horizon is $T = 400$ and the terminal condition is constructed by using T-period stationary economy in all experiments.

6 Numerical analysis of nonstationary and unbalanced growth applications

We provide a collection of numerical examples that illustrate how EFP can be used for calibrating, solving, estimating and simulating nonstationary problems. Our examples include models with an unbalanced growth path, expected and unexpected technology shocks, seasonal adjustments and deterministically changing volatility of productivity shocks, as well as an example of calibrating and estimating parameters in an unbalanced growth model using the data on the U.S. economy. These applications do not allow for stationary Markov equilibria and hence, cannot be studied with conventional solution methods. The model's parameterization, time horizon and terminal condition differ across applications – we describe them separately for each application considered.

6.1 Application 1: An unbalanced growth model with a CES production function and capital-augmenting technological progress

The previous section focused on a nonstationary growth model that can be converted into a stationary model and that can be studied with conventional solution methods. We now consider a nonstationary model that cannot be converted into a stationary model and cannot be studied with conventional methods. Namely, we assume a constant elasticity of substitution (CES) production function, and we allow for both labor and capital augmenting technological progresses,

$$F(k_t, \ell_t) = \left[\alpha (A_{k,t} k_t)^v + (1-\alpha)(A_{\ell,t}\ell_t)^v\right]^{1/v}, \tag{19}$$

where $A_{k,t} = A_{k,0} g_{A_k}^t$; $A_{\ell,t} = A_{\ell,0} g_{A_\ell}^t$; $v \leq 1$; $\alpha \in (0,1)$; g_{A_k} and g_{A_ℓ} are the rates of capital and labour augmenting technological progresses, respectively. We assume that labor is supplied inelastically and normalize it to one $\ell_t = 1$ for all t, and we denote the corresponding production function by $f(k_t) \equiv F(k_t, 1)$. The model with capital augmenting technological progress does not satisfy the assumptions in King, Plosser and Rebelo (1988) and does not admit a balanced growth path.

The assumption of capital augmenting technological progress is advocated in the literature on directed technical change. Acemoglu (2002) points out that in most cases, technical change does not apply to the same fixed factors of production all the time but is endogenously directed to those factors of production that can give the largest improvement in the efficiency of production.[5] An implication of this argument that is relevant for our analysis is that technical change can be directed either to capital or to labor or other production factors depending on a particular case.

Furthermore, Acemoglu (2003) explicitly incorporates capital augmenting technological progress into a deterministic model of endogenous technical change with both labor and capital augmenting innovations. Empirical estimates of the growth rates of the capital augmenting technical change can be found in, e.g., Klump, Mc Adam and Willman (2007), and León-Ledesma

[5] Namely, endogenous technical change is biased toward a relatively more scarce factor when the elasticity of substitution is low (because this factor is relatively more expensive); however, it is biased toward a relatively more abundant factor when the elasticity of substitution is high (because technologies using such a factor have a larger market).

León-Ledesma, Mc Adam and Wilman (2015). Below, we show how the model with capital augmenting technological progress can be studied by using EFP.

6.1.1 A growth path for a nonstationary economy

Our first goal is to define a growth path around which the sequence of grids will be centered. For constructing the growth path, we shut down uncertainty by assuming $z_t = 1$ for all t (similar to what we do for a model with balanced growth) and we rewrite the model's equations in the way that is convenient for identifying the path.

First, the Euler equation of period t, evaluated on the steady state path, is

$$1 = \beta \left[\frac{u'(c^*_{t+1})}{u'(c^*_t)} (1 - \delta + f' \left\{ \alpha A^v_{k,t+1} (k^*_{t+1})^{v-1} \left[\alpha (A_{k,t+1} k^*_{t+1})^v + (1-\alpha) A^v_{\ell,t+1} \right]^{(1-v)/v} \right\} \right],$$

where c^*_t and k^*_t are the variables on the growth path. From the last equation, we express k^*_{t+1} as

$$k^*_{t+1} = (1-\alpha)^{1/v} \frac{A_{\ell,t+1}}{A_{k,t+1}} \left[\left(\frac{(g_{u',t+1})^{-1} - \beta + \delta\beta}{\alpha\beta \cdot A_{k,t+1}} \right)^{v/(1-v)} - \alpha \right]^{1/v}, \tag{20}$$

where $g_{u',t+1} \equiv \frac{u'(c^*_{t+1})}{u'(c^*_t)}$ follows from the budget constraints (2) for t and $t+1$:

$$g_{u',t+1} = \frac{u' \left[(1-\delta) k^*_{t+1} + \left[\alpha (A_{k,t+1} k^*_{t+1})^v + (1-\alpha) A^v_{\ell,t+1} \right]^{1/v} - k^*_{t+2} \right]}{u' \left[(1-\delta) k^*_t + \left[\alpha (A_{k,t} k^*_t)^v + (1-\alpha) A^v_{\ell,t} \right]^{1/v} - k^*_{t+1} \right]}. \tag{21}$$

Thus, we obtain a system of $T-1$ equations (20) with $T+1$ unknowns $k^*_0, ..., k^*_{T+1}$. This system does not have a unique solution unless we impose two additional restrictions.

6.1.2 Identifying restrictions on initial and terminal conditions

There are many possible ways to impose identifying restrictions on the solution of system (20), (21). In this specific application, we restrict the initial and terminal capital stocks, k^*_0 and k^*_{T+1}. Namely, we restrict k^*_0 by assuming that the capital growth rate is the same in the first two periods $\frac{k^*_1}{k^*_0} = \frac{k^*_2}{k^*_1}$, and we restrict k^*_{T+1} by assuming such a growth rate is the same in the last two periods $\frac{k^*_T}{k^*_{T-1}} = \frac{k^*_{T+1}}{k^*_T}$. This pins down the initial and terminal capital stocks on the growth path in terms of $(k^*_1, ..., k^*_T)$,

$$k^*_0 = \frac{(k^*_1)^2}{k^*_2} \qquad \text{and} \qquad k^*_{T+1} = \frac{(k^*_T)^2}{k^*_{T-1}}. \tag{22}$$

The model satisfies the assumptions of King, Plosser and Rebelo (1988) if there is only labor augmenting technological progress, i.e., $A_{\ell,t}$ grows at a constant, exogenously given rate g_{A_ℓ} and $A_{k,t} = A_k$ for all t. In this special case, the model has a balanced growth path on which all variables grow at a constant rate g_{A_ℓ} and this is in particular true for initial and terminal periods, i.e., condition (22) is satisfied exactly.

In the case of capital augmenting technological progress, the growth rate of endogenous variables changes over time in an unbalanced manner even if we assume that $A_{k,t}$ grows at a constant, exogenously given growth rate g_{A_k} and $A_{\ell,t} = A_\ell$ for all t. By imposing two additional restrictions in (22), we solve for $k_0^*, ..., k_{T+1}^*$ satisfying (20), (21). In our applications, changes in the growth path $k_0^*, ..., k_{T+1}^*$ had only a minor effect on the quality of the approximations. This is because a specific growth path does not identify the solution itself but only a set of points in which the Smolyak grids are centered. Centering a grid in a slightly different point will not significantly affect the properties of solution in a typical application. The assumption in (22) can be modified if needed.

6.1.3 Results of numerical experiments

For numerical experiments, we assume $T = 260$, $\gamma = 1$, $\alpha = 0.36$, $\beta = 0.99$, $\delta = 0.025$, $\rho = 0.95$, $\sigma_\varepsilon = 0.01$, $\upsilon = -0.42$; the last value is taken in line with Antrás (2004) who estimated the elasticity of substitution between capital and labor to be in the range $[0.641, 0.892]$ that corresponds to $\upsilon \in [-0.12, -0.56]$. We solve two models: the model with labor augmenting progress parameterized by $A_{\ell,0} = 1.1130$, $g_{A_\ell} = 1.00153$ and $A_{k,0} = g_{A_k} = 1$ and the model with capital augmenting progress parameterized by $A_{k,0} = 1$, $g_{A_k} = 0.9867$ and $A_{\ell,0} = g_{A_\ell} = 1$. (The parameters $A_{\ell,0}$, g_{A_ℓ}, $A_{k,0}$, g_{A_k} for both models are chosen to approximately match the initial and terminal capital stocks for time-series solutions of both models).

Figure 4 plots the time-series solutions for models with labour and capital augmenting progresses, as well as their growth paths.

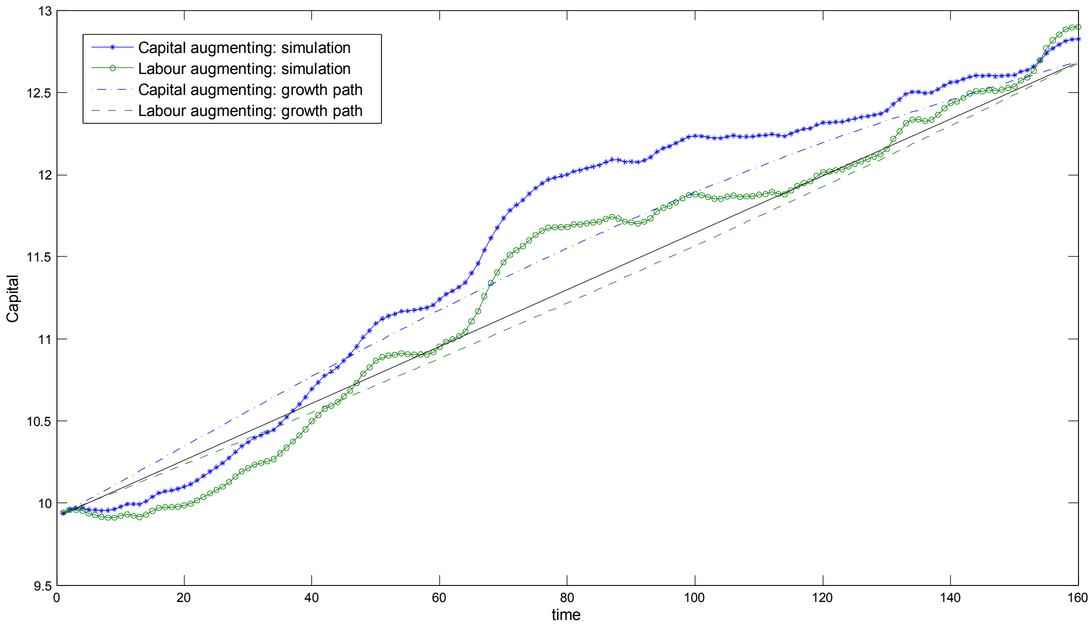

Figure 4: Technological progress in the model with the CES production

The solution of the model with labor augmenting technological progress is typical for a balanced growth model. There is an exponential growth path with a constant growth rate and cyclical fluctuations around the growth path. (In the figure, the growth path in the model with labor augmenting technological progress is situated below the linear growth path shown by a solid line). In turn, the solution of the model with capital augmenting progress has a pronounced concave growth pattern that shows that the rate of return to capital decreases as the economy grows (In the figure, the growth path in the model with capital augmenting technological progress is situated above the linear growth path shown by a solid line). Finally, the cyclical properties of both models look similar (provided that growth is detrended). These are novel results since the properties of stochastic growth models with capital augmenting progress are not studied yet in the literature (to the best of our knowledge).

6.2 Application 2: A nonstationary model with a parameter shift

The recent literature on regime switches addresses the critiques of naive solution methods of Cooley, Leroy and Raymon (1984) and provides a logically consistent way of modeling unanticipated regime switches. Specifically, this literature assumes that agents solve maximization problems in which regime are possible, and thus, they can adequately react to regime changes in simulation as implied by their decision functions; see Sims and Zha (2006), Davig and Leeper (2007, 2009), Farmer, Waggoner, and Zha (2011), Foerster, Rubio-Ramírez, Waggoner and Zha (2013) and Zhong (2015), among others.

The above literature assumes that the regimes come at random, drawn from a stationary probability distribution. However, there are real-world situations in which parameter shifts are nonrecurrent and anticipated by agents in advance, e.g., seasonal changes, presidential elections with anticipated outcome, forward-looking policy announcements, anticipated technological advances, etc. A prominent example of an anticipated shock is an accession of new members to the European Union that was announced many years in advance and that resulted in quantitatively-important anticipatory effects; see Garmel, Maliar and Maliar (2008) for a discussion and a quantitative assessment of such effects in a three-country general equilibrium model.

Schmitt-Grohé and Uribe (2012) propose a computational approach that allows to deal with anticipated parameter shifts of fixed time horizons in the context of stationary Markov models (the parameter shifts systematically occur, for example, each fourth or each eighth periods). However, if the anticipated parameter shifts are either nonrecurrent and do not have fixed anticipation horizons, the model does not admit stationary Markov solutions and cannot be studied using conventional solution methods. However, EFP can solve models with such anticipated shocks. As an example, we show how to solve a model with anticipated technology shocks, and we compare the solution produced by EFP to naive solutions in which shocks are unanticipated.

6.2.1 Anticipated technology shocks

The idea that anticipated shocks play an important role in business cycle fluctuations goes back to Pigou (1927). The literature that advocates the importance of anticipated shocks for aggregate fluctuations includes, e.g., Cochrane (1994), Beaudry and Portier (2006), and Schmitt-Grohé and Uribe (2012).

We consider a version of the model (1), (2), (17) and (18) in which the technology level A_t can take two values, $\underline{A} = 1$ (low) and $\bar{A} = 1.2$ (high). A special case of this setup is a model in which $\underline{A}$ and $\bar{A}$ are unanticipated and randomly drawn from a given probability distribution. Such a model has a stationary Markov solution that can be studied using the approaches described in the literature on regime switches, e.g., Davig and Leeper (2007, 2009).

In contrast, we focus on the case when the regime switches are both nonrecurrent and anticipated by the agent from the beginning. As an example, we consider a scenario when the economy starts with $\underline{A}$ at $t = 0$, switches to $\bar{A}$ at $t' = 250$ and then switches back to $\underline{A}$ at $t'' = 550$ (instead, we could have considered any other scenario for technology levels). We show the technology profile in the upper panel of Figure 5.

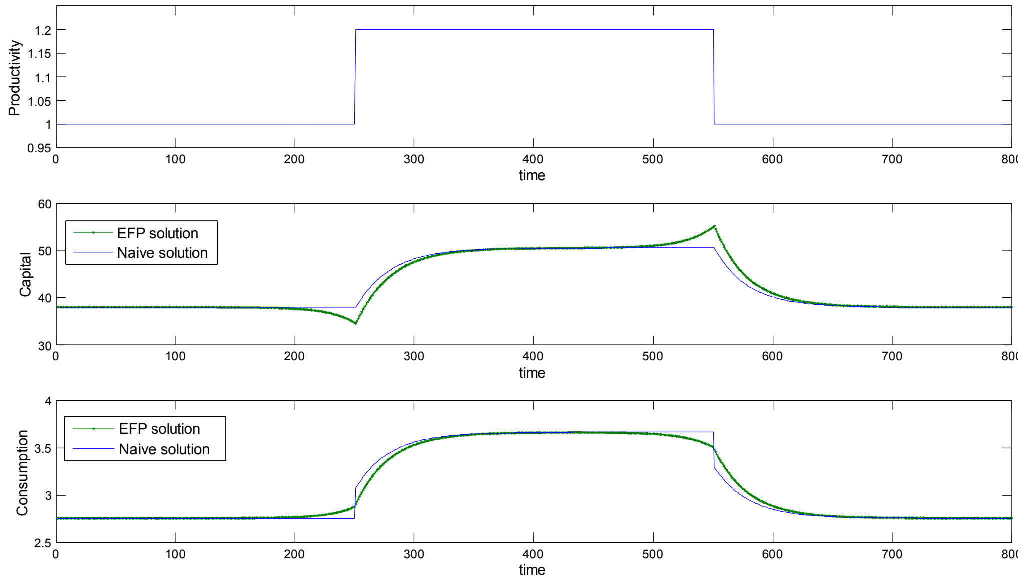

Figure 5. Anticipated versus unanticipated technology shocks

The other parameters are $T = 900$, $\gamma = 1$, $\alpha = 0.36$, $\beta = 0.99$, $\delta = 0.025$, $\rho = 0.95$, $\sigma_\varepsilon = 0.01$.

The agent solves the utility-maximization problem at $t = 0$ given the technology profile. The implementation of the EFP method for this case is similar to the one with technological progress studied in the previous section (as initial and terminal values of the growth path, k_0^* and k_{T+1}^*, we use a steady state of the model with $\underline{A}$). In the case of the naive solution, shocks are unexpected. We construct two stationary naive solutions under $\underline{A}$ and $\bar{A}$. The agent follows the first solution until the first switch at $t' = 250$, then the agent follows the second solution until the second switch at $t'' = 550$ and finally, the agent goes back to the first solution for the rest of the simulation.

The two time-series solutions for capital and consumption are shown, respectively, in the middle and lower panels of Figure 5. In simulation, we set $z_t = 1$ for all t to make the anticipatory effects more visible. Remarkably, in the solution with the expected regime switches, we observe a strong anticipatory effect: about 50 periods before the switch from $\underline{A}$ and $\bar{A}$ takes place, the agent starts gradually increasing her consumption and to decrease her capital stock in order to bring some part of the benefits from future technology growth to present.

When the technology switch actually occurs, it has only a minor effect on consumption. (The tendencies reverse when there is a switch from $\bar{A}$ to $\underline{A}$). In contrast, consumption-smoothing anticipatory effects are absent for the naive solution. Here, unexpected technology shocks lead to large jumps in consumption in the exact moment of technology switches. The difference in the solutions is quantitatively significant under our empirically plausible parameter choice. Finally, in the Appendix E, we plot the simulated solution by considering both deterministic technology switches and stochastic productivity shocks following an AR(1) process (37); see Figure 10. Anticipatory effects are well pronounced in those experiments as well.

6.2.2 A model with seasonal changes

One empirically-relevant application that our EFP framework can deal with is a growth model with seasonal changes. An important role of seasonal fluctuations in the total variation in aggregate economic variables is well documented in the literature; see, e.g., Barsky and Miron (1989). Ignoring seasonality when estimating dynamic stochastic general equilibrium models may lead to substantial errors in the estimated parameters; see, e.g., Saijo (2013).

Two approaches have been proposed in the literature to model seasonality. Hansen and Sargent (1993, 2013) characterize seasonality in terms of the spectral density of variables. They assume that seasonality comes either from seasonality in exogenous shock processes (with spectral peaks at seasonal frequencies) or from propagation mechanisms determined by preferences and technology (e.g., seasonal habit persistence) or from seasonal periodicity in the parameters of the preferences and technologies; in these cases, the optimal decision rules are periodic. Second, Christiano and Todd (2002) develop a model in which exogenous shocks contain deterministic seasonal dummies and in which investment process is period-specific (an investment project requires four quarters to complete, and current-period total investment depends on the projects started in the current and three previous periods); to solve such a model, they linearize around the model's seasonally varying steady state growth path and solve for four distinct decision rules.

We now show how to solve economic models with seasonal changes by using EFP. As an example, we assume that every forth period, A_t takes a high value $\overline{A}$, and the rest of the periods, it takes a low value $\underline{A}$; for example, this pattern can be observed in a country on a seacoast in which there is a high productivity season in summer. Thus, we obtain the following sequence of technology levels: $\underline{A}, \underline{A}, \underline{A}, \overline{A}, \underline{A}, \underline{A}, \underline{A}, \overline{A}, ...$ In addition to the seasonal changes, the agent faces the conventional productivity shocks (37), so that the resulting path for the productivity level is given by a composition of expected seasonal changes in A_t and unexpected stochastic changes in productivity levels given by a stationary autoregressive process. The parameters are the same as in the previous model except that we use $\gamma = 2$, $\beta = 0.97$, $\underline{A} = 0.98$ and $\overline{A} = 1.06$ (these parameters are fixed for expositional convenience). To construct the growth path for the EFP method, we set both the initial and terminal conditions at $3/4\underline{k}^* + 1/4\overline{k}^*$, where $\underline{k}^*$ and $\overline{k}^*$ are the steady states of capital in the models with $\underline{A}$ and $\overline{A}$, respectively. In Figure 6, we plot time series for productivity, capital and consumption (we normalize the initial values of all series to one).

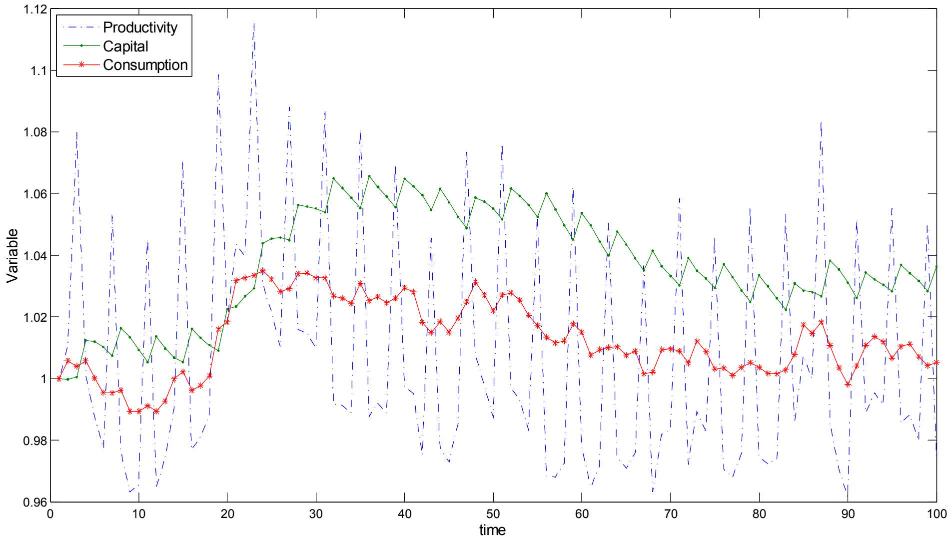

Figure 6. Seasonality

An interesting finding in Figure 6 is that the size of seasonal consumption and capital fluctuations is very small compared to the size of seasonal productivity fluctuations. A consumption-smoothing agent knows that the seasonal shock is temporary and that it does not pay to react much on the impact of such a shock. Instead, the agent adjusts her capital and consumption to take advantage of seasonal productivity growth on average, as permanent consumption hypothesis suggests. A magnitude of seasonal fluctuations in the model's variables is far larger and comparable in size to seasonal productivity fluctuations in a naive solution in which the seasonal shocks are unexpected as naive agents would fail to take into account anticipatory effects (we do not provide the naive solution to avoid a clutter).

6.3 Application 3: A nonstationary model with a parameter drift

A class of models with parameter drifting is another interesting and empirically relevant type of economic applications that are characterized by nonstationary solutions. There is ample empirical evidence in favor of parameter drifting, see, e.g., Clarida, Galí and Gertler (2000), Lubick and Schorfheide (2004), Cogley and Sargent (2005), Goodfriend and King (2009), Canova (2009). The assumption of parameter drifting is advocated in Galí (2006). The previous literature focuses on economic models with stationary Markov equilibria by assuming that the model's parameters follow a stationary autoregressive process; see, e.g., Fernández-Villaverde and Rubio-Ramírez (2007), Fernández-Villaverde, Guerrón-Quintana and Rubio-Ramírez (2010). However, if the model's parameters follow a pattern with a pronounced time trend, the equilibrium decision rules change from one period to another and the conventional solution methods are not applicable. Below, we show how to use EFP to solve an example of the model with parameter drift that includes time trends.

6.3.1 A growth model with a productivity drift

We consider a scenario that is similar to the one in Application 2, however, we now assume that technology does not switch to a higher/lower level in one period but increases/decreases gradually. To be specific, the level of technology is low, $\underline{A}$, for the first 200 periods; it increases linearly to a high level, $\overline{A}$, for the next 100 periods; it stays constant for the following 300 periods; it decreases linearly back to a low level, $\underline{A}$, for 200 periods and finally, it stays there for the remaining period of time. The above productivity profile is shown in Figure 7.

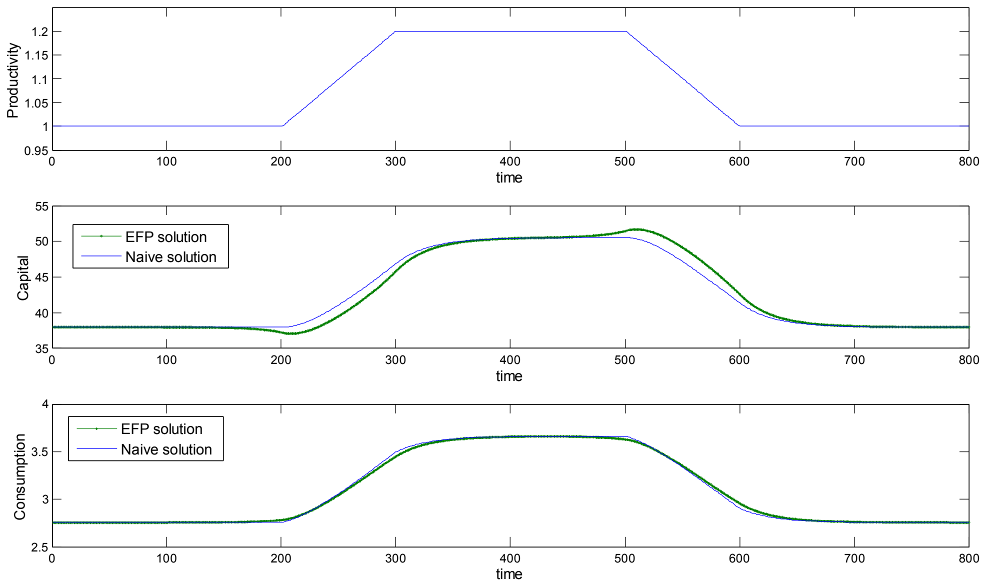

Figure 7. Technology drift

To calibrate the model, we use the same parameters as in Application 2.

We plot the EFP time-series solution of the model with a parameter drift in the middle and lower panel of Figure 7. For a comparison, we also provide a naive solution in which shocks are always unanticipated. To produce the naive solution, we solve a stationary model 100 times under each level of technology that occurs in the parameter drift, and we jump from one stationary naive solution to another after each technology change. Again, to simulate the solution, we set $z_t = 1$ for all t for a better visibility of anticipatory effects.

Similar to Application 2, we observe well-pronounced smoothing of consumption at the cost of anticipatory adjustments of capital. In particular, the consumption path with an expected parameter drift is smoother than the one in the naive solution in those places where the parameter shift begins / ends and we observe the kink. In the Appendix E, we provide a plot of the simulated solution with both deterministic productivity shifts and stochastic productivity shocks; see Figure 11. Again, anticipatory effects are well pronounced in that case as well.

6.3.2 Example of a parameter drift: diminishing volatility

A large body of recent literature documents the importance of degree of uncertainty for the business cycle. This literature argues that volatility changes over time. They model volatility (e.g., standard deviation of the productivity level) as a stochastic process or as a regime switch; see, e.g., Bloom (2009), Fernández-Villaverde and Rubio-Ramírez (2010), Fernández-Villaverde, Guerrón-Quintana and Rubio-Ramírez (2010). The literature normally assumes that the standard deviation of exogenous shocks either follows a Markov process or experiences recurring Markov regime switches. In the latter case, volatility can be treated as an additional state variable, and in the former case, the regime is an additional state variable; in both cases, it is possible to cast the model with changing volatility into the conventional stationary framework.

However, there is evidence that the volatility has a well pronounced time trend, for example, Mc Connel and Pérez-Quiros (2000) document a monotone structural decline in the volatility of real GDP growth in the U.S. economy. Blanchard and Simon (2001) find a nonmonotone pattern of the decline in the U.S. GDP volatility: there was a steady decline in the volatility from the 1950s to 1970, then there was a stationary pattern and finally, there was another decline in the late 1980s and the 1990s. Stock and Watson (2003) find a sharp reduction in volatility of U.S. GDP growth in the first quarter of 1984. This kind of evidence cannot be reconciled in a model in which stochastic volatility follows a standard AR(1) process with time-invariant parameters. We show how to use EFP to study a model in which the volatility has both a stochastic and deterministic components.

We specifically consider the standard neoclassical stochastic growth model, modified to include a diminishing volatility of the productivity shock:

$$\ln z_t = \rho \ln z_{t-1} + \sigma_t \varepsilon_t, \qquad \sigma_t = \frac{B}{t^{\rho_\sigma}}, \quad \varepsilon_t \sim \mathcal{N}(0,1), \tag{23}$$

where B is a scaling parameter, and ρ_σ is a parameter that governs the volatility of z_t. The standard deviation of the productivity shock $B\sigma/t^{\rho_\sigma}$ decreases over time, reaching zero in the limit, $\lim_{t\to\infty} \frac{B\sigma}{t^{\rho_\sigma}} = 0$.

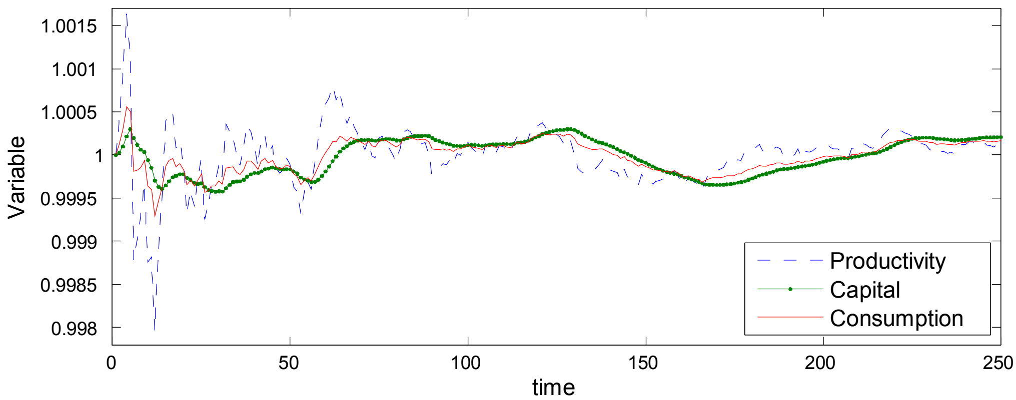

Figure 8. Diminishing volatility

In our numerical example, we use $T = 500$, $\gamma = 1$, $\alpha = 0.36$, $\beta = 0.99$, $\delta = 0.025$, $\rho = 0.95$, $\sigma_\varepsilon = 0.01$, $B = 1$ and $\rho_\sigma = 1.05$. To solve this model, we use EFP in which we build a grid around the deterministic steady state value of capital. In Figure 8, we plot a sequence of simulated productivity levels; as we see, initially, there are large productivity fluctuations but gradually, these fluctuations become smaller. As expected, fluctuations in capital and consumption also decrease in amplitude in response to diminishing volatility.

6.4 Application 4: Calibrating a growth model with a parameter drift to unbalanced U.S. data

There is a large group of econometric methods that estimate and calibrate economic models by constructing numerical solutions explicitly, including simulated method of moments (e.g., Canova (2007)); Bayesian estimation method (e.g., Smets and Wouters (2003), and Del Negro, Schorfheide, Smets and Wouters (2007)); and maximum likelihood method (e.g., Fernández-Villaverde and Rubio-Ramírez (2007)). Normally, the related literature imposes restrictions on the model that lead to a balanced growth path, converts the model into stationary model and solves it for stationary Markov equilibrium by using conventional methods.

However, there are two potential problems with this approach. First, the restrictions that are necessary to impose for balanced growth might not be the empirically-relevant ones. For example, we might want to analyze a model with nonhomothetic utility and production functions, several kinds of technical progress and parameter shifts and drifts. However, any deviation from the restrictions in King, Plosser and Rebelo (1988) destroys the property of balanced growth and hence, destroys the conventional Markov stationary equilibria. Second, the real world data are not always consistent with the assumption of balanced growth, in particular, different variables might grow at different and possibly time-varying rates. In this section, we illustrate how EFP can be used to calibrate and estimate parameters in an unbalanced growth model by using the data on U.S. economy.

6.4.1 Time series to match

We took macroeconomic data on the U.S. economy from the webpages of the Bureau of Economic Analysis and the Federal Reserve Bank of St. Louis (namely, the data on capital and investment come from the former data base, while the data on the remaining time series, as well as that on the implicit price deflator, come from the latter data base); the sample spans over the period 1964:Q1 - 2011:Q4. Investment is defined as nonresidential and residential private fixed investment. Consumption is defined as a sum of nondurables and services. Capital is given by a sum of fixed assets and durables; capital series are annual (in contrast to the other series which are quarterly); we interpolate annual series of capital to get quarterly series using spline interpolation. Output is obtained as a sum of consumption and investment. We deflate the constructed variables with the corresponding implicit price deflator and we convert them in per capita terms by dividing them by the series of the total population.

6.4.2 The model with a depreciation rate drift

While the constructed data are grossly consistent with Kaldor's (1961) facts, we still observe visible differences in growth rates across variables. We do not test whether or not such differences in growth rates are statistically significant but formulate and estimate an unbalanced growth model in which different variables can grow at differing rates. We specifically extend the model (1)–(3) to include time-varying depreciation rate of capital,

$$\max_{\{c_t,k_{t+1}\}_{t=0,\ldots,\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{24}$$

$$\text{s.t. } c_t + k_{t+1} = A_t z_t k_t^{\alpha} + (1 - d_t\delta_t)\, k_t, \tag{25}$$

$$\ln \delta_t = \rho_\delta \ln \delta_{t-1} + \varepsilon_{\delta,t}, \qquad \varepsilon_{\delta,t} \sim \mathcal{N}\left(0, \sigma_{\varepsilon_d}^2\right), \tag{26}$$

$$\ln z_t = \rho_z \ln z_{t-1} + \varepsilon_{z,t}, \qquad \varepsilon_{z,t} \sim \mathcal{N}\left(0, \sigma_{\varepsilon_z}^2\right), \tag{27}$$

where $d_t\delta_t$ stands for a time-varying depreciation rate with d_t being a trend component of depreciation, $d_t = d_0 g_d^t$, and δ_t being a stochastic shock to depreciation. Our assumption of a time trend in depreciation rate is consistent with the data of the Bureau of Economic Analysis. In particular, the aggregate depreciation rate changes over time because the compositional of aggregate capital changes over time even if depreciation rates of each type of capital are constant; see Karabarbounis and Brent (2014). In turn, shocks to depreciation rate can result from the economic obsolescence of capital and are studied in, e.g., Liu, Waggoner and Zha (2011), Gourio (2012) and Zhong (2015), in particular, this literature argues that a shock to the capital depreciation rate plays an important role in accounting for the business cycle fluctuations.

6.4.3 Calibration and estimation of the model's parameters

To identify the model's parameters, we formulate the following set of restrictions

$$A_t z_t = \frac{y_t}{k_t^{\alpha}}, \tag{28}$$

$$d_t\delta_t = \frac{i_t}{k_t} - \frac{k_{t+1} - k_t}{k_t}, \tag{29}$$

$$\frac{1}{\beta} = \frac{1}{T}\sum_{t=1}^{T} \frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}} \left[1 - d_{t+1}\delta_{t+1} + \alpha A_{t+1} z_{t+1} k_{t+1}^{\alpha-1}\right]. \tag{30}$$

We set $\gamma = 1$ and we search for α that matches best the growth rates of variables in the data. First, given some α, we construct $A_t z_t$ using (28) , and we estimate the parameters ρ_z, $\sigma_{\varepsilon_z}^2$, g_A in the process for productivity $z_t = z_{t-1}^{\rho_z} \exp(\varepsilon_{z,t})$ using a linear regression method. To identify a growing and cycle components, A_t and z_t, respectively, we assume $z_0 = 1$. Second, we construct the data on $d_t\delta_t$ using (29), and we estimate the parameters ρ_δ, $\sigma_{\varepsilon_\delta}^2$, g_d in the process for productivity $\delta_t = \delta_{t-1}^{\rho_\delta} \exp(\varepsilon_{\delta,t})$ using a linear regression. Again, to separate growth and cycles, d_t and δ_t, respectively, we assume $\delta_0 = 1$. Finally, we calibrate the discount factor by using the Euler equation (30).

Our estimation-calibration procedure gives the following values of parameters: $\beta = 0.9013$, $\rho_z = 0.9890$, $\sigma_{\varepsilon_z} = 0.0054$, $g_A = 1.002$, $\rho_\delta = 0.9538$, $\sigma_{\varepsilon_\delta} = 0.0381$ and $g_d = 1.002$. We observe a considerable positive growth rate in the depreciation rate $g_d = 1.002$. Furthermore, we find that the best fit of our criteria for the growth rate is obtained under $\alpha = 0.7$. This value for the capital share in output is larger than is typically used in the business cycle literature, however, it is roughly in line with the recent finding of Karabarbounis and Neiman (2014) that labor shares gradually declined over time; the implied gross capital shares reach 0.55.

We know that on the tail, the EFP solution will depend on a specific terminal condition used and may be insufficiently accurate. To deal with this issue, we extrapolate the data for 80 periods forward, using the growth rates that we estimate from the data on consumption, capital, output, and investment under the assumption of exponential growth. We implement EFP to match the initial and terminal conditions in the extrapolated data, i.e., we use $T = \tau + 80$. To identify the growth path in our unbalanced growth model, we use assumption (22). We construct a sequence of growing Smolyak grids. There are three state variables (k_t, z_t, δ_t) in this application and the corresponding second-level Smolyak grid consists of 25 multidimensional grid points. After we compute the EFP solution, we simulate the model using the sequence of shocks reconstructed from the data.

6.4.4 Fitted time series

Figure 9 presents the simulated time-series solution for capital, output, investment and consumption; for comparison, we also provide the corresponding time series from the data. To appreciate the differences in growth rates, we scaled all four panels to have the same percentage change in y.

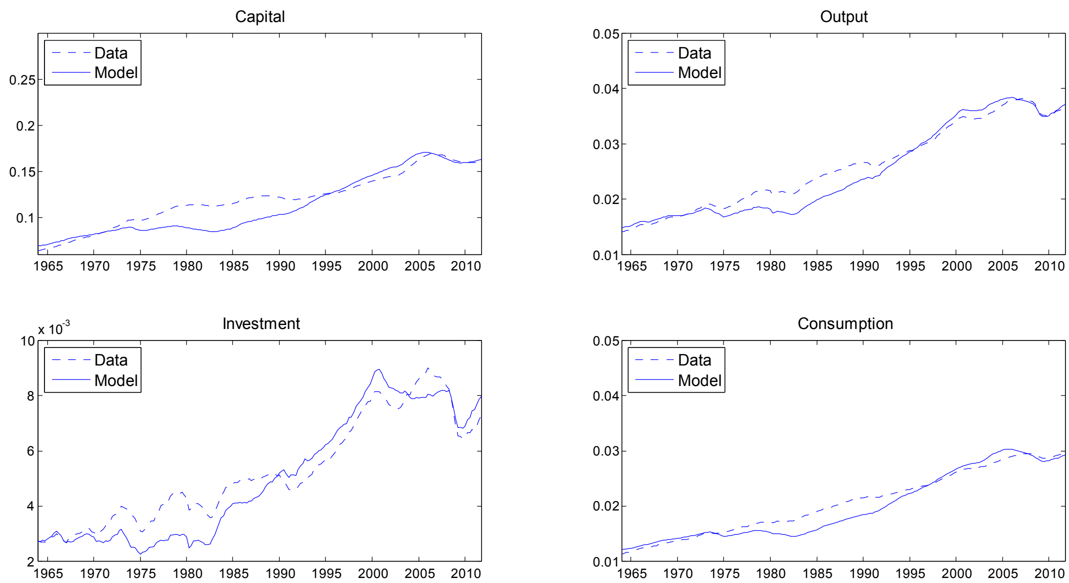

Figure 9. Matching nonstationary macroeconomic data on the U.S. economy

First of all, we can visually appreciate the nonstationarity in the data: investment grows considerably faster than other variables. With the assumption of time-varying depreciation rate, the model (24)–(27) can closely reproduce the growth rates of all model's variables.

The main goal of this application is not to advocate the role of time-varying depreciation rate or some specific estimation and calibration technique. Rather, we would like to illustrate how estimation and calibration of the parameters can be carried out in the context of a nested fixed-point problem without assuming stationarity and balanced growth. Similar to the depreciation rate, we could have made all other parameters time dependent, including the discount factor β, the share of capital in production α and the parameters of the process for the productivity level (27). Furthermore, our simple estimation-calibration technique can be replaced by more sophisticated econometric techniques such as maximum likelihood, simulated method of moments, etc.

7 Conclusion

Stationary Markov dynamic economic models are a dominant framework in recent economic literature. A shortcoming of this framework is that it generally restricts the structural parameters of economic models to be constant, and it restricts the behavior patterns to be time invariant. In this paper, we construct a more flexible class of nonstationary Markov models that allows for time-varying preferences, technology and laws of motions for exogenous variables. We propose EFP framework for solving, calibrating, simulating and estimating of such models. EFP enables us to analyze economic models that do not admit stationary Markov equilibria and that cannot be studied with conventional solution methods. Literally, EFP makes it possible to analyze a unique historical path of real-world economies.

Our analysis can be extended in three possible directions: First, our numerical results are produced by an Euler equation version of EFP that finds a path of decision functions to satisfy a sequence of Euler equations. It is also of interest to explore the performance of an analogous value-iterative version of EFP that first constructs a value function for some remote period T and then constructs a path of time-varying value functions that matches the given terminal value function.

Second, we build EFP on global approximation techniques that construct decision functions on a sequence of domains covered with Smolyak grids. It is of much practical interest to develop also a version of EFP that builds on local perturbation techniques. The conventional EP method of Fair and Taylor (1983) is incorporated in the Dynare software platform, and possibly, a perturbation-based version of the proposed EFP framework can be added there as well.[6]

Finally, the goal of the present paper is to introduce the EFP methodology, and the standard optimal growth model is a convenient framework for this goal. In our ongoing research, we use EFP framework for analyzing challenging nonstationary and unbalanced growth applications that go beyond the standard growth model. In one project, we incorporate a production function with capital-skill complementarity in a general equilibrium unbalanced growth model with the aim of explaining unbalanced growth patterns in the U.S. economy data. Our analysis builds around production functions studied in Katz and Murphy (1992) and Krusell, Ohanian, Ríos-Rull and Violante (2000). In another project, we augment a new Keynesian model, studied in Maliar and Maliar (2015) and extend epsilon-distingushable-set and cluster-grid methods, to include nonrecurrent switches in monetary policy regimes, and we attempt to reproduce the sequence of events during the Great Recession. Our explanation follows the analysis of Taylor (2012), who documents a historical sequence of events that led to the recent 2008-2013 Great Recession and argues that the recession is likely to be related to changes in monetary policy; also, see Del Negro, Giannoni and Schorfheide (2015) for related evidence. In our other

[6] For more details on the Dynare software, see http://www.dynare.org and the Dynare reference manual by Adjemian, Bastani, Juillard, Mihoubi, Perendia, Ratto, and Villemot (2011).

project, we investigate the role of anticipated stabilization policies in nonstationary economies with default risk by combining the EFP framework with the value-iterative envelope condition method, introduced in Maliar and Maliar (2013) and Arellano, Maliar, Maliar and Tsyrennikov (2014). These are just three examples but many interesting and empirically relevant questions can be addressed by using EFP framework.

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Appendices to "A Tractable Framework for Analyzing Nonstationary and Unbalanced Growth Models"

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Appendix A. Asymptotic convergence of T-period stationary economy to nonstationary economy

This section elaborates the proof of Theorem 2 (turnpike theorem) formulated in Section 3.2, specifically, it shows that the optimal program of the T-period stationary economy converges to the optimal program of the nonstationary economy (1)–(3) as $T \to \infty$. The proof relies on three lemmas presented in Appendices A1-A3. In Appendix A1, we construct a limit program of a finite horizon economy with a terminal condition $k_T = 0$. In Appendix A2, we show that the optimal program of the T-period stationary economy, constructed in Section 3.1, converges to the same limit program as does the finite horizon economy with a zero terminal condition $k_T = 0$. In Appendix A3, we show that the limit program of the finite horizon economy with a zero terminal condition $k_T = 0$ is also an optimal program for the infinite horizon nonstationary economy (1)–(3). Finally, in Appendix A4, we combine the results of Appendices A1-A3 to establish the claim of Theorem 2. Our construction relies on mathematical tools developed in Majumdar and Zilcha (1987), Mitra and Nyarko (1991), Joshi (1997). We use the convention that equalities and inequalities hold almost everywhere (a.e.) except for a set of measure zero.

Appendix A1. Limit program of finite horizon economy with a zero terminal capital

In this section, we consider a finite horizon version of the economy (1)–(3) with a given terminal condition for capital k_T. Specifically, we assume that the agent solves

$$\max_{\{c_t,k_{t+1}\}_{t=0}^{T}} E_0 \left[\sum_{t=0}^{T} \beta^t u_t (c_t) \right] \tag{31}$$

$$\text{s.t. (2), (3),} \tag{32}$$

where initial condition (k_0, z_0) and terminal condition k_T are given. We first define feasible programs for the finite horizon economy.

Definition A1 *(Feasible programs in the finite horizon economy).* A feasible program in the finite horizon economy is a pair of adapted (i.e., $\mathcal{F}_t$ measurable for all t) processes $\{c_t, k_t\}_{t=0}^{T}$

such that given initial condition k_0 and a partial history $h_T = (\varepsilon_0, ..., \varepsilon_T)$, they reach a given terminal condition k_T at T, satisfy $c_t \geq 0$, $k_t \geq 0$ and (2), (3) for all $t = 1, ...T$.

In this section, we focus on a finite horizon economy that reaches a zero terminal condition, $k_T = 0$, at T. We denote by $\Im^{T,0}$ a set of all finite horizon feasible programs from given initial capital k_0 and given partial history $h_T \equiv (\varepsilon_0, ..., \varepsilon_T)$ that attain given $k_T = 0$ at T. We next introduce the concept of solution for the finite horizon model.

Definition A2 *(Optimal program in the finite horizon model).* A feasible finite horizon program $\left\{c_t^{T,0}, k_t^{T,0}\right\}_{t=0}^{T} \in \Im^{T,0}$ is called optimal if

$$E_0 \left[\sum_{t=0}^{T} \beta^t \left\{ u_t(c_t^{T,0}) - u_t(c_t) \right\} \right] \geq 0 \tag{A1}$$

for every feasible process $\{c_t, k_t\}_{t=0}^{T} \in \Im^{T,0}$.

The existence result for the finite horizon version of the economy (31), (32) with a zero terminal condition is established in the literature. Namely, under Assumptions 1-9, there exists an optimal program $\left\{c_t^{T,0}, k_t^{T,0}\right\}_{t=0}^{T} \in \Im^{T,0}$ and it is both interior and unique. The existence of the optimal program can be shown by using either a Bellman equation approach (see Mitra and Nyarko (1991), Theorem 3.1) or an Euler equation approach (see Majumdar and Zilcha (1987), Theorems 1 and 2).

We next show that under terminal condition $k_T^{T,0} = k_T = 0$, the optimal program in the finite horizon economy (31), (32) has a well-defined limit.

Lemma 1. *A finite horizon optimal program* $\left\{c_t^{T,0}, k_t^{T,0}\right\}_{t=0}^{T} \in \Im^{T,0}$ *with a zero terminal condition* $k_T^{T,0} = 0$ *converges to a limit program* $\left\{c_t^{\lim}, k_t^{\lim}\right\}_{t=0}^{\infty}$ *when* $T \to \infty$, *i.e.,*

$$k_t^{\lim} \equiv \lim_{T \to \infty} k_t^{T,0} \quad \text{and} \quad c_t^{\lim} \equiv \lim_{T \to \infty} c_t^{T,0}, \quad \text{for } t = 0, 1, ... \tag{A2}$$

Proof. The existence of the limit program follows by three arguments:

i) Extending time horizon from T to $T+1$ increases T-period capital of the finite horizon optimal program, i.e., $k_T^{T+1,0} > k_T^{T,0}$. To see this, note that the model with time horizon T has zero (terminal) capital $k_T^{T,0} = 0$ at T. When time horizon is extended from T to $T+1$, the model has zero (terminal) capital $k_{T+1}^{T+1,0} = 0$ at $T+1$ but it has strictly positive capital $k_T^{T+1,0} > 0$ at T; this follows by the Inada conditions–Assumption 4.

ii) The optimal program for the finite horizon economy has the following property of monotonicity with respect to the terminal condition: if $\{c_t', k_t'\}_{t=0}^{T}$ and $\{c_t'', k_t''\}_{t=0}^{T}$ are two optimal programs for the finite horizon economy with terminal conditions $\kappa' < \kappa''$, then the respective optimal capital choices have the same ranking in each period, i.e., $k_t' \leq k_t''$ for all $t = 1, ...T$. This monotonicity result follows by either Bellman equation programming techniques (see Mitra and Nyarko (1991, Theorem 3.2 and Corollary 3.3)) or Euler equation programming techniques (see Majumdar and Zilcha (1987, Theorem 3)) or lattice programming techniques (see Hopenhayn and Prescott (1992)); see also Joshi (1997, Theorem 1) for generalizations of these results to nonconvex economies. Hence, the stochastic process $\left\{k_t^{T,0}\right\}_{t=0}^{T}$ shifts up (weakly) in a pointwise manner when T increases to $T+1$, i.e., $k_t^{T+1,0} \geq k_t^{T,0}$ for $t \geq 0$.

iii) By construction, the capital program from the optimal program $\left\{c_t^{T,0}, k_t^{T,0}\right\}_{t=0}^{T}$ is bounded from above by the capital accumulation process $\{0, k_t^{\max}\}_{t=0}^{T}$ defined in (8), i.e., $k_t^{T,0} \leq k_t^{\max} < \infty$ for $t \geq 0$. A sequence that is bounded and monotone is known to have a well-defined limit. ■

Appendix A2. Limit program of the T-period stationary economy

We now show that the optimal program of the T-period stationary economy, introduced in Section 3.1, converges to the same limit program (A2) as the optimal program of the finite horizon economy (31), (32) with a zero terminal condition. We denote by $\Im^T$ a set of all feasible finite horizon programs that attains a terminal condition of the T-period stationary economy. (We assume the same initial capital (k_0, z_0) and the same partial history $h_T \equiv (\varepsilon_0, ..., \varepsilon_T)$ as those fixed for the finite horizon economy (31), (32)).

Lemma 2. *The optimal program of the T-period stationary economy $\{c_t^T, k_t^T\}_{t=0}^{T} \in \Im^T$ converges to a unique limit program $\{c_t^{\lim}, k_t^{\lim}\}_{t=0}^{\infty} \in \Im^{\infty}$ defined in (A2) as $T \to \infty$ i.e., for all $t \geq 0$*

$$k_t^{\lim} \equiv \lim_{T\to\infty} k_t^T \quad \text{and} \quad c_t^{\lim} \equiv \lim_{T\to\infty} c_t^T. \tag{A3}$$

Proof. The proof of the lemma follows by six arguments.

i). Observe that, by Assumptions 1-8, the optimal program of the T-period stationary economy has a positive capital stock $k_T^T > 0$ at T (since the terminal capital is generated by the capital decision function of a stationary version of the model), while for the optimal program $\left\{c_t^{T,0}, k_t^{T,0}\right\}_{t=0}^{T} \in \Im^{T,0}$ of the finite horizon economy, it is zero by definition, $k_T^{T,0} = 0$.

ii). The property of monotonicity with respect to terminal condition implies that if $k_T^T > k_T^{T,0}$, then $k_t^T \geq k_t^{T,0}$ for all $t = 1, ..., T$; see our discussion in ii). of the proof to Lemma 1.

iii). Let us fix some $\tau \in \{1, ..., T\}$. We show that up to period τ, the optimal program $\{c_t^T, k_t^T\}_{t=0}^{\tau}$ does not give higher expected utility than $\left\{c_t^{T,0}, k_t^{T,0}\right\}_{t=0}^{\tau}$, i.e.,

$$E_0\left[\sum_{t=0}^{\tau} \beta^t \left\{u_t\left(c_t^T\right) - u_t(c_t^{T,0})\right\}\right] \leq 0. \tag{A4}$$

Toward contradiction, assume that it does, i.e.,

$$E_0\left[\sum_{t=0}^{\tau} \beta^t \left\{u_t\left(c_t^T\right) - u_t(c_t^{T,0})\right\}\right] > 0. \tag{A5}$$

Then, consider a new process $\{c_t', k_t'\}_{t=0}^{\tau}$ that follows $\{c_t^T, k_t^T\}_{t=0}^{T} \in \Im^T$ up to period $\tau - 1$ and that drops down at τ to match $k_\tau^{T,0}$ of the finite horizon program $\{c_t^T, k_t^T\}_{t=0}^{T} \in \Im^{T,0}$, i.e.,

$\{c'_t, k'_t\}_{t=0}^{\tau} \equiv \left\{c_t^T, k_t^T\right\}_{t=0}^{\tau-1} \cup \left\{c_\tau^T + k_\tau^T - k_\tau^{T,0}, k_\tau^{T,0}\right\}$. By monotonicity ii). we have $k_\tau^T - k_\tau^{T,0} \geq 0$, so that

$$E_0\left[\sum_{t=0}^{\tau} \beta^t \left\{u_t\left(c'_t\right) - u_t\left(c_t^T\right)\right\}\right] =$$

$$= E_0\left[\beta^\tau \left\{u_t\left(c_\tau^T + k_\tau^T - k_\tau^{T,0}\right) - u_t\left(c_\tau^T\right)\right\}\right] \geq 0, \quad \text{(A6)}$$

where the last inequality follows by Assumption 2 of strictly increasing u_t.

iv). By construction $\{c'_t, k'_t\}_{t=0}^{\tau}$ and $\left\{c_t^{T,0}, k_t^{T,0}\right\}_{t=0}^{\tau}$ reach the same capital $k_\tau^{T,0}$ at τ. Let us extend the program $\{c'_t, k'_t\}_{t=0}^{\tau}$ to T by assuming that it follows the process $\left\{c_t^{T,0}, k_t^{T,0}\right\}_{t=0}^{T}$ from the period $\tau+1$ up to T, i.e., $\{c'_t, k'_t\}_{t=\tau+1}^{T} \equiv \left\{c_t^{T,0}, k_t^{T,0}\right\}_{t=\tau+1}^{T}$. Then, we have

$$E_0\left[\sum_{t=0}^{T} \beta^t \left\{u_t\left(c'_t\right) - u_t(c_t^{T,0})\right\}\right] = E_0\left[\sum_{t=0}^{\tau} \beta^t \left\{u_t\left(c'_t\right) - u_t(c_t^{T,0})\right\}\right]$$

$$\geq E_0\left[\sum_{t=0}^{\tau} \beta^t \left\{u_t\left(c_t^T\right) - u_t\left(c_t^{T,0}\right)\right\}\right] > 0, \quad \text{(A7)}$$

where the last two inequalities follow by result (A6) and assumption (A5), respectively. Thus, we obtain a contradiction: The constructed program $\{c'_t, k'_t\}_{t=0}^{T} \in \Im^{T,0}$ is feasible in the finite horizon economy with a zero terminal condition, $k'_T = 0$, and it gives strictly higher expected utility than the optimal program $\left\{c_t^{T,0}, k_t^{T,0}\right\}_{t=0}^{T} \in \Im^{T,0}$ in that economy.

v). Holding τ fixed, we compute the limit of (A4) by letting T go to infinity:

$$\lim_{T\to\infty} E_0\left[\sum_{t=0}^{\tau} \beta^t \left\{u_t\left(c_t^T\right) - u_t(c_t^{T,0})\right\}\right] =$$

$$\lim_{T\to\infty} E_0\left[\sum_{t=0}^{\tau} \beta^t u_t\left(c_t^T\right)\right] - E_0\left[\sum_{t=0}^{\tau} \beta^t u_t\left(c_t^{\lim}\right)\right] \leq 0. \quad \text{(A8)}$$

vi). The last inequality implies that for any $\tau \geq 1$, the limit program $\left\{c_t^{\lim}, k_t^{\lim}\right\}_{t=0}^{\infty} \in \Im^\infty$ of the finite horizon economy $\left\{c_t^{T,0}, k_t^{T,0}\right\}_{t=0}^{T} \in \Im^{T,0}$ with a zero terminal condition $k_T^{T,0} = 0$ gives at least as high expected utility as the optimal limit program $\left\{c_t^T, k_t^T\right\}_{t=0}^{T} \in \Im^T$ of the T-period stationary economy. Since Assumptions 1-8 imply that the optimal program is unique, we conclude that $\left\{c_t^{\lim}, k_t^{\lim}\right\}_{t=0}^{\infty} \in \Im^\infty$ defined in (A2) is a unique limit of the optimal program $\left\{c_t^T, k_t^T\right\}_{t=0}^{T} \in \Im^T$ of the T-period stationary economy. ■

Appendix A3. Convergence of finite horizon economy to infinite horizon economy

We now show a connection between the optimal programs of the finite horizon and infinite horizon economies. Namely, we show that the finite horizon economy (31), (32) with a zero terminal condition $k_T^{T,0} = 0$ converges to the nonstationary infinite horizon economy (1)–(3) as $T \to \infty$.

Lemma 3. *The limit program* $\left\{c_t^{\lim}, k_t^{\lim}\right\}_{t=0}^{\infty}$ *is a unique optimal program* $\{c_t^{\infty}, k_t^{\infty}\}_{t=0}^{\infty} \in \Im^{\infty}$ *in the infinite horizon nonstationary economy (1)–(3).*

Proof. We prove this lemma by contradiction. We use the arguments that are similar to those used in the proof of Lemma 2.

i). Toward contradiction, assume that $\left\{c_t^{\lim}, k_t^{\lim}\right\}_{t=0}^{\infty}$ is not an optimal program of the infinite horizon economy $\{c_t^{\infty}, k_t^{\infty}\}_{t=0}^{\infty} \in \Im^{\infty}$. By definition of limit, there exists a real number $\epsilon > 0$ and a subsequence of natural numbers $\{T_1, T_2, ...\} \subseteq \{0, 1, ...\}$ such that $\{c_t^{\infty}, k_t^{\infty}\}_{t=0}^{\infty} \in \Im^{\infty}$ gives strictly higher expected utility than the limit program of the finite horizon economy $\left\{c_t^{\lim}, k_t^{\lim}\right\}_{t=0}^{\infty}$, i.e.,

$$E_0\left[\sum_{t=0}^{T_n} \beta^t \left\{u_t\left(c_t^{\infty}\right) - u_t(c_t^{\lim})\right\}\right] > \epsilon \text{ for all } T_n \in \{T_1, T_2, ...\}\,. \tag{A9}$$

ii). Let us fix some $T_n \in \{T_1, T_2, ...\}$ and consider any finite $T \geq T_n$. Assumptions 1-8 imply that $k_T^{\infty} > 0$ while $k_T^{T,0} = 0$ by definition of the finite horizon economy with a zero terminal condition. The monotonicity of the optimal program with respect to a terminal condition implies that if $k_T^{\infty} > k_T^{T,0}$, then $k_t^{\infty} \geq k_t^{T,0}$ for all $t = 1, ..., T$; see our discussion in ii). of the proof of Lemma 1.

iii). Following the arguments in iii). of the proof of Lemma 2, we can show that up to period T_n, the optimal program $\{c_t^{\infty}, k_t^{\infty}\}_{t=0}^{T_n}$ does not give higher expected utility than $\left\{c_t^{T,0}, k_t^{T,0}\right\}_{t=0}^{T_n}$, i.e.,

$$E_0\left[\sum_{t=0}^{T_n} \beta^t \left\{u_t\left(c_t^{\infty}\right) - u_t(c_t^{T,0})\right\}\right] \leq 0 \text{ for all } T_n. \tag{A10}$$

iv). Holding T_n fixed, we compute the limit of (A10) by letting T go to infinity:

$$\lim_{T\to\infty} E_0\left[\sum_{t=0}^{T_n} \beta^t \left\{u_t\left(c_t^{\infty}\right) - u_t(c_t^{T,0})\right\}\right]$$
$$= E_0\left[\sum_{t=0}^{T_n} \beta^t u_t\left(c_t^{\infty}\right) - \beta^t u_t\left(c_t^{\lim}\right)\right] \leq 0 \text{ for all } T_n. \tag{A11}$$

However, result (A11) contradicts to our assumption in (A9).

v). We conclude that for any subsequence $\{T_1, T_2, ...\} \subseteq \{0, 1, ...\}$, we have

$$E_0\left[\sum_{t=0}^{T_n} \beta^t \left\{u_t\left(c_t^{\infty}\right) - u_t(c_t^{\lim})\right\}\right] \leq 0 \text{ for all } T_n. \tag{A12}$$

However, under Assumptions 1-8, the optimal program $\{c_t^{\infty}, k_t^{\infty}\}_{t=0}^{\infty} \in \Im^{\infty}$ is unique, and hence, it must be that $\{c_t^{\infty}, k_t^{\infty}\}_{t=0}^{\infty}$ coincides with $\left\{c_t^{\lim}, k_t^{\lim}\right\}_{t=0}^{\infty}$ for all $t \geq 0$. ■

Appendix A4. Proof to the turnpike theorem

We now combine the results of Lemmas 1-3 together into a turnpike-style theorem to show the convergence of the optimal program of the T-period stationary economy to that of the infinite horizon nonstationary economy. To be specific, Lemma 1 shows that the optimal program of the finite horizon economy with a zero terminal condition $\left\{c_t^{T,0}, k_t^{T,0}\right\}_{t=0}^{T} \in \Im^{T,0}$ converges to the limit program $\{c_t^{\lim}, k_t^{\lim}\}_{t=0}^{\infty}$. Lemma 2 shows that the optimal program of the T-period stationary economy $\{c_t^T, k_t^T\}_{t=0}^{T}$ also converges to the same limit program $\{c_t^{\lim}, k_t^{\lim}\}_{t=0}^{\infty}$. Finally, Lemma 3 shows that the limit program of the finite horizon economies $\{c_t^{\lim}, k_t^{\lim}\}_{t=0}^{\infty}$ is optimal in the nonstationary infinite horizon economy. Then, it must be the case that the limit optimal program of the T-period stationary economy $\{c_t^T, k_t^T\}_{t=0}^{T}$ is optimal in the infinite horizon nonstationary economy. This argument is formalized below.

Proof to Theorem 2 (turnpike theorem). The proof follows by definition of limit and Lemmas 1-3. Let us fix a real number $\varepsilon > 0$ and a natural number τ such that $1 \leq \tau < \infty$.

i). Lemma 1 shows that $\left\{c_t^{T,0}, k_t^{T,0}\right\}_{t=0}^{T} \in \Im^{T,0}$ converges to a limit program $\{c_t^{\lim}, k_t^{\lim}\}_{t=0}^{\infty}$ as $T \to \infty$. Then, definition of limit implies that there exists $T_1 > 0$ such that $\left|k_t^{T,0} - k_t^{\lim}\right| < \frac{\varepsilon}{3}$ for $t = 0, ..., \tau$.

ii). Lemma 2 implies that the finite horizon problem of the T-period stationary economy $\{c_t^T, k_t^T\}_{t=0}^{T}$ also converges to limit program $\{c_t^{\lim}, k_t^{\lim}\}_{t=0}^{\infty}$ as $T \to \infty$. Then, there exists $T_2 > 0$ such that $\left|k_t^{\lim} - k_t^T\right| < \frac{\varepsilon}{3}$ for $t = 0, ..., \tau$.

iii). Lemma 3 implies the program $\left\{c_t^{T,0}, k_t^{T,0}\right\}_{t=0}^{T} \in \Im^{T,0}$ converges to the infinite horizon optimal program $\{c_t^{\infty}, k_t^{\infty}\}_{t=0}^{\infty}$ as $T \to \infty$. Then, there exists $T_3 > 0$ such that $\left|k_t^{T,0} - k_t^{\infty}\right| < \frac{\varepsilon}{3}$ for $t = 0, ..., \tau$.

iv). Then, the triangular inequality implies

$$\begin{aligned}\left|k_t^T - k_t^{\infty}\right| &= \left|k_t^T - k_t^{\lim} + k_t^{\lim} - k_t^{T,0} + k_t^{T,0} - k_t^{\infty}\right| \\ &\leq \left|k_t^T - k_t^{\lim}\right| + \left|k_t^{\lim} - k_t^{T,0}\right| + \left|k_t^{T,0} - k_t^{\infty}\right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,\end{aligned}$$

for $T \geq T(\varepsilon, \tau) \equiv \max\{T_1, T_2, T_3\}$. ■

Remark A1. Our proof of the turnpike theorem addresses a technical issue that does not arise in the literature that focuses on finite horizon economies with a zero terminal condition; see, e.g., Majumdar and Zilcha (1987), Mitra and Nyarko (1991) and Joshi (1997). Their construction relies on the fact that the optimal program of the finite horizon economy is always pointwise below the optimal program of the infinite horizon economy, i.e., $k_t^T \leq k_t^{\infty}$, for $t = 1, ..., \tau$, and it gives strictly higher expected utility up to T than does the infinite horizon optimal program (because excess capital can be consumed at terminal period T). This argument does not directly applies to our T-period stationary economy: our finite horizon program can be either below or above the infinite horizon program depending on a specific T-period terminal condition; see the experiments with terminal conditions κ' and κ'' in Figure 1, respectively. Our proof addresses this issue by constructing in Lemma 2 a separate limit program for the T-period stationary economy.

Remark A2. We also proved a similar turnpike theorem for a more general version of the economy (1)–(3) (proofs are not provided). First, we relax the assumption of Markov structure of the stochastic process (3) (i.e., we consider a general stochastic environment that satisfies only a weak assumptions of measurability); and second, we relax the assumption that the terminal condition comes from the T-period stationary economy (i.e., we consider an arbitrary terminal condition k_T). To save on space, we do not include this more general turnpike theorem in the paper but limit ourselves to the nonstationary Markov setup that is actually studied.

Appendix B. Implementation of EFP

In this section, we describe the implementation of the EFP method used to produce the numerical results in the main text.

Algorithm 1 (implementation). Extended function path.

The goal of EFP.

EFP is aimed at approximating a solution of a nonstationary model during the first τ periods, i.e., it finds approximating functions $\left(\widehat{K}_0, ..., \widehat{K}_\tau\right)$ such that $\widehat{K}_t \approx K_t$ for $t = 1, ...\tau$, where K_t and $\widehat{K}_t$ are a t-period true capital function and its parametric approximation, respectively.

Step 0. Initialization.

a. Choose time horizon $T \gg \tau$ for constructing T-period stationary economy.
b. Construct a deterministic path $\{z_t^*\}_{t=0}^T$ for exogenous state variable $\{z_t\}_{t=0}^T$ satisfying
$z_{t+1}^* = \varphi_t\left(z_t^*, E_t\left[\varepsilon_{t+1}\right]\right)$ for $t = 0, .., T$.
c. Construct a deterministic path $\{k_t^*\}_{t=0}^T$ for endogenous state variable $\{k_t\}_{t=0}^T$ satisfying
$u_t'(c_t^*) = \beta u_t'(c_{t+1}^*)(1 - \delta + f_{t+1}'\left(k_{t+1}^*, z_{t+1}^*\right))$.
$c_t^* + k_{t+1}^* = (1-\delta) k_t^* + f_t\left(k_t^*, z_t^*\right)$ for $t = 0, .., T$.
d. For $t = 0, ..., T$:
Construct a grid $\{(k_{m,t}, z_{m,t})\}_{m=1}^M$ centered at (k_t^*, z_t^*).
Choose integration nodes, $\varepsilon_{j,t}$, and weights, $\omega_{j,t}$ for $j = 1, ..., J$.
Construct future shocks $z_{m,j,t}' = \varphi_t\left(z_{m,t}, \varepsilon_{j,t}\right)$.
e. Write a t-period discretized system of the optimality conditions:

i). $u_t'(c_{m,t}) = \beta \sum_{j=1}^{J} \omega_{j,t}\left[u_t'(c_{m,j,t}')\left\{1 - \delta + f_{t+1}\left(k_{m,t}', z_{m,j,t}'\right)\right\}\right]$

ii). $c_{m,t} + k_{m,t}' = (1-\delta) k_{m,t} + f_t\left(k_{m,t}, z_{m,t}\right)$

iii). $c_{m,j,t}' + k_{m,j,t}'' = (1-\delta) k_{m,t}' + f_{t+1}\left(k_{m,t}', z_{m,j,t}'\right)$

iv). $k_{m,t}' = \widehat{K}_t\left(k_{m,t}, z_{m,t}\right)$ and $k_{m,j,t}'' = \widehat{K}_{t+1}\left(k_{m,t}', z_{m,j,t}'\right)$.

d. Assume that the model becomes stationary at T.

Step 1. Solving the T-period stationary model.

Find $\widehat{K}_T = \widehat{K}_{T+1}$ that approximately solves the system i).-iv). on the grid for the T-period stationary economy $f_{T+1} = f_T$, $u_{T+1} = u_T$, $\varphi_{T+1} = \varphi_T$.

Step 2. Solving for a function path for $t = 0, 1, ..., T-1$.

a. Construct the function path $\left(\widehat{K}_0, ..., \widehat{K}_{T-1}, \widehat{K}_T\right)$ that approximately solves the system i).-iv) for each $t = 0, ..., T$ and that matches the given terminal function $\widehat{K}_T$ constructed in Step 1.

The EFP solution:

Use $\left(\widehat{K}_0, ..., \widehat{K}_\tau\right)$ as an approximation to $(K_0, ..., K_\tau)$ and discard the remaining $T - \tau$ functions.

The EFP method is more expensive than conventional solution methods for stationary models because decision functions must be constructed not just once but for T periods. We implement EFP in the way that keeps its cost relatively low: First, to approximate decision functions, we use a version of the Smolyak (sparse) grid technique. Specifically, we use a version of the Smolyak method that combines a Smolyak grid with ordinary polynomials for approximating functions off the grid. This method is described in Maliar, Maliar and Judd (2011) who find it to be sufficiently accurate in the context of a similar growth model, namely, unit-free residuals in the model's equations do not exceed 0.01% on a stochastic simulation of 10,000 observations). For this version of the Smolyak method, the polynomial coefficients are overdetermined, for example, in a 2-dimensional case, we have 13 points in a second-level Smolyak grid, and we have only six coefficients in second-degree ordinary polynomial. Hence, we identify the coefficients using a least-squares regression; we use an SVD decomposition, to enhance numerical stability; see Judd, Maliar and Maliar (2011) for a discussion of this and other numerically stable approximation methods. We do not construct the Smolyak grid within a hypercube normalized to $[-1, 1]^2$, like do Smolyak methods that rely on Chebyshev polynomials used in, e.g., Krueger and Kubler (2004) and Judd, Maliar, Maliar and Valero (2014). Instead, we construct a sequence of Smolyak grids around actual steady state and thus, the hypercube, in which the Smolyak grid is constructed, grows over time as shown in Figure 1.

Second, to approximate expectation functions, we use Gauss-Hermite quadrature rule with 10 integration nodes. However, a comparison analysis in Judd, Maliar and Maliar (2011) shows that for models with smooth decision functions like ours, the number of integration nodes plays only a minor role in the properties of the solution, for example, the results will be the same up to six digits of precision if instead of ten integration nodes we use just two nodes or a simple linear monomial rule (a two-node Gauss-Hermite quadrature rule is equivalent to a linear monomial integration rule for the two-dimensional case). However, simulation-based Monte-Carlo-style integration methods produce very inaccurate approximations for integrals and are not considered in this paper; see Judd, Maliar and Maliar (2011) for discussion.

Third, to solve for the coefficients of decision functions, we use a simple derivative-free fixed-point iteration method in line with Gauss-Jacobi iteration. Let us re-write the Euler equation i). constructed in the initialization step of the algorithm by pre-multiplying both sides by t-period capital

$$\widehat{k}'_{m,t} = \beta \sum_{j=1}^{J} \varepsilon_{j,t} \left[\frac{u'_t(c'_{m,j,t})}{u'_t(c_{m,t})} \left\{ 1 - \delta + f_{t+1}\left(k'_{m,t} k^*_{t+1}, z'_{m,j,t} z^*_{t+1} \right) \right\} \right] k'_{m,t}. \tag{33}$$

We use different notation, $k'_{m,t}$ and $\widehat{k}'_{m,t}$, for t-period capital in the left and right side of (33), respectively, in order to describe our fixed-point iteration method. Namely, we substitute $k'_{m,t}$ in the right side of (33) and in the constraints ii). and iii). in the initialization step to compute $c_{m,t}$ and $c'_{m,j,t}$, respectively, and we obtain a new set of values of the capital function on the grid $\widehat{k}'_{m,t}$ in the left side. We iterate on these steps until convergence.

Our approximation functions $\widehat{K}_t$ are ordinary polynomial functions characterized by a time-dependent vector of parameters b_t, i.e., $\widehat{K}_t = \widehat{K}(\cdot; b_t)$. So, operationally, the iteration is performed not on the grid values $k'_{m,t}$ and $\widehat{k}'_{m,t}$ but on the coefficients of the approximation functions. The iteration procedure differs in Steps 1 and 2.

In Step 1, we construct a solution to T-period stationary economy. For iteration i, we fix some initial vector of coefficients b, compute $k'_{m,T+1} = \widehat{K}(k_{m,T}, z_{m,T}; b)$, find $c_{m,T}$ and $c'_{m,j,T}$

to satisfy constraints ii) and iii), respectively and find $\widehat{k}'_{m,T+1}$ from the Euler equation i). We run a regression of $\widehat{k}'_{m,T+1}$ on $\widehat{K}\left(k_{m,T}, z_{m,T}; \cdot\right)$ in order to re-estimate the coefficients $\widehat{b}$ and we compute the coefficients for iteration $i+1$ as a weighted average, i.e., $b^{(i+1)} = (1-\xi)\, b^{(i)} + \xi \widehat{b}^{(i)}$, where $\xi \in (0,1)$ is a damping parameter (typically, $\xi = 0.05$). We use partial updating instead of full updating $\xi = 1$ because fixed-point iteration can be numerically unstable and using partial updating enhances numerical stability; see Maliar, Maliar and Judd (2011). This kind of fixed-point iterations are used by numerical methods that solve for equilibrium in conventional stationary Markov economies; see e.g., Judd, Maliar and Maliar (2011), Judd, Maliar, Maliar and Valero (2014).

In Step 2, we iterate on the path for the polynomial coefficients using Gauss-Jacobi style iterations in line with Fair and Taylor (1983). Specifically, on iteration j, we take a path for the coefficients vectors $\left\{b_1^{(j)}, ..., b_T^{(j)}\right\}$, compute the corresponding path for capital quantities using $k'_{m,t} = \widehat{K}_t\left(k_{m,t}, z_{m,t}; b_t^{(j)}\right)$, and find a path for consumption quantities $c_{m,t}$ and $c'_{m,j,t}$ from constraints ii) and iii), respectively, for $t = 0, ..., T$. Substitute these quantities in the right side of a sequence of Euler equations for $t = 0, ..., T$ to obtain a new path for capital quantities in the left side of the Euler equation $\widehat{k}'_{m,t}$ for $t = 0, ..., T-1$. Run $T-1$ regressions of $\widehat{k}'_{m,t}$ on polynomial functional forms $\widehat{K}_t\left(k_{m,t}, z_{m,t}; b_t\right)$ for $t = 0, ..., T-1$ to construct a new path for the coefficients $\left\{\widehat{b}_0^{(j)}, ..., \widehat{b}_{T-1}^{(j)}\right\}$. Compute the path of the coefficients for iteration $j+1$ as a weighted average, i.e., $b_t^{(j+1)} = (1-\xi)\, b_t^{(j)} + \xi \widehat{b}_t^{(j)}$, $t = 0, ..., T-1$, where $\xi \in (0,1)$ is a damping parameter which we again typically set at $\xi = 0.05$. (Observe that this iteration procedure changes all the coefficients on the path except of the last one $b_T^{(j)} \equiv b$, which is a given terminal conditions that we computed in Step 1 from T-period stationary economy).

In fact, the problem of constructing a path for function coefficients is similar to the problem of constructing a path for variables: in both cases, we need to solve a large system of nonlinear equations. The difference is that under EFP, the arguments of this system are not variables but parameters of the approximating functions. Instead of Gauss-Jacobi style iteration on path, we can use Gauss-Siedel fixed-point iteration (shooting), Newton-style solvers or any other technique that can solve a system of nonlinear equations; see Lipton, Poterba, Sachs and Summers (1980), Atolia and Buffie (2009a,b), Heer and Maußner (2010), and Grüne, Semmler and Stieler (2013) for examples of such techniques.

Appendix C. Fair and Taylor's (1983) method

This appendix describes the version of Fair and Taylor's (1983) method used to produce the results in the main text. We illustrate this method in the context of the growth model (1)–(3) (we assume $\delta = 1$ and $u(c) = \ln(c)$ for expository convenience). The Euler equation and budget constraint, respectively, are:

$$\frac{1}{c_t} = \beta E_t\left[\frac{1}{c_{t+1}}\left(1 - \delta + z_{t+1} f'(k_{t+1})\right)\right],$$
$$c_t + k_{t+1} = (1-\delta)\, k_t + z_t f\left(k_t\right).$$

Combine the above two conditions to get

$$k_{t+1} = z_t f(k_t) - \left[E_t \left(\frac{\beta z_{t+1} f'(k_{t+1}))}{z_{t+1} f(k_{t+1}) - k_{t+2}} \right) \right]^{-1} \approx$$

$$z_t f(k_t) - \frac{z^e_{t+1} f(k_{t+1}) - k_{t+2}}{\beta z^e_{t+1} f'(k_{t+1}))}, \tag{34}$$

where the expectation function $z^e_{t+1} = E_t[z_{t+1}]$ is approximated as implied by certainty equivalence assumption (16). For example, for the standard AR(1) process for productivity levels (18), this approximation means that for each t, we have $z^e_{t+1} = E[z_{t+1}] = z^\rho_t E[\exp(\varepsilon_t)]$, where $\varepsilon_t \sim N(0, \sigma^2)$. To solve for the path of variables, Fair and Taylor (1983) use a derivative-free iteration in line with Gauss-Jacobi method :

Algorithm 2. Extended path (EP) framework by Fair and Taylor (1983).

The goal of EP framework of Fair and Taylor (1983).
EFP is aimed at approximating a path for variables satisfying the model's equations during the first τ periods, i.e., it finds $\widehat{k}_0, ..., \widehat{k}_\tau$ such that $\left\Vert k_t - \widehat{k}_t \right\Vert < \varepsilon$ for $t = 1, ...\tau$, where $\varepsilon > 0$ is target accuracy, $\Vert\cdot\Vert$ is an absolute value, and k_t and $\widehat{k}_t$ are the t-period true capital stocks and their approximation, respectively.
Step 0. Initialization.
a. Choose time horizon $T \gg \tau$ and terminal condition k_{T+1}. b. Construct and fix a sequence of shocks $\{z_t\}_{t=0,...,T}$. c. Construct and fix $\{z^e_{t+1}\}_{t=0,...,T}$ such that $z^e_{t+1} = E_t[z_{t+1}]$ for all t. d. Guess an equilibrium path $\left\{k^{(1)}_t\right\}_{t=1,...,T'}$ for iteration $j = 1$. e. Write a t-period system of the optimality conditions in the form: $k_{t+1} = z_t f(k_t) - \frac{z^e_{t+1} f(k_{t+1}) - k_{t+2}}{\beta z^e_{t+1} f'(k_{t+1}))}$.
Step 1. Solving for a path using Gauss-Jacobi method.
a. Substitute a path $\left\{k^{(j)}_t\right\}_{t=1,...,T'}$ into the right side of (34) to find $k^{(j+1)}_{t+1} = z_t f\left(k^{(j)}_t\right) - \frac{z^e_{t+1} f\left(k^{(j)}_{t+1}\right) - k^{(j)}_{t+2}}{\beta z^e_{t+1} f'(k^{(j)}_{t+1}))}, \qquad t = 1, ..., T$ b. End iteration if the convergence is achieved $\left\vert k^{(j+1)}_{t+1} - k^{(j)}_{t+1} \right\vert <$ tolerance level. Otherwise, increase j by 1 and repeat Step 1.
The EP solution:
Use the first τ constructed values $k_0, ..., k_\tau$ as an approximation to the true solution $k_t, ..., k_\tau$ and discard the last $T - \tau$ values.

In the original version of their EP method, Fair and Taylor (1983) use $\tau = 1$, i.e., they keep only the first element $\widehat{k}_1$ from the constructed path $\left(\widehat{k}_1, ..., \widehat{k}_T\right)$ and disregard the rest of the path; then, they solve for a new path $\left(\widehat{k}_1, ..., \widehat{k}_{T+1}\right)$ starting from $\widehat{k}_1$ and ending in a given $\widehat{k}_{T+1}$ and store $\widehat{k}_2$, disregarding the rest of the path; and they advance forward until the path of the given length τ is constructed. T is chosen so that further its extensions do not affect the solution in the initial period of the path. For instance, to find a solution $\widehat{k}_1$, Fair and Taylor (1983) solve the model several times under $T+1$, $T+2$, $T+3, ...$ and check that $\widehat{k}_1$ remains the same (up to a given degree of precision). Finally, it is also possible to use Fair and Taylor's (1983) method under larger values of τ such as $\tau = 100$; in this respect, Fair and Taylor's (1983) method is similar to EFP.

As is typical for fixed-point-iteration style methods, Gauss-Jacobi iteration may fail to converge. To deal with this issue, Fair and Taylor (1983) use damping, namely, they update the path over iteration only by a small amount $k_{t+1}^{(j+1)} = \xi k_{t+1}^{(j+1)} + (1-\xi) k_{t+1}^{(j)}$ where $\xi \in (0,1)$ is a small number close to zero (e.g., 0.01).

Steps 1a and *1b* of Fair and Taylor's (1983) method are called Type I and Type II iterations and are analogous to *Step 2* of the EFP method when the sequence of the decision functions is constructed. The extension of path is called Type III iteration and gives the name to Fair and Taylor (1983) method.

In our examples, we implement Fair and Taylor's (1983) method using a conventional Newton style numerical solver instead of Gauss-Jacobi iteration; a similar implementation is used in Heer and Maußner (2010). The cost of Fair and Taylor's (1983) method can depend considerably on a specific solver used and can be very high (as we need to solve a system of equations with hundreds of unknowns numerically). In our simple examples, a Newton-style solver was sufficiently fast and reliable. In more complicated models, we are typically unable to derive closed-form laws of motion for the state variables, and derivative-free fixed-point iteration advocated in Fair and Taylor (1983) can be a better alternative.

Appendix D. Solving the test model using the associated stationary model

We first convert the nonstationary model (1), (2), (17), (18) with labor augmenting technological progress into a stationary model using the standard change of variables $\widehat{c}_t = c_t/A_t$ and $\widehat{k}_t = k_t/A_t$. This leads us to the following model

$$\max_{\left\{\widehat{k}_{t+1}, \widehat{c}_t\right\}_{t=0,\dots,\infty}} E_0 \sum_{t=0}^{\infty} (\beta^*)^t \frac{\widehat{c}_t^{1-\gamma}}{1-\gamma} \tag{35}$$

$$\text{s.t. } \widehat{c}_t + g_A \widehat{k}_{t+1} = (1-\delta)\,\widehat{k}_t + z_t \widehat{k}_t^{\alpha}, \tag{36}$$

$$\ln z_{t+1} = \rho_t \ln z_t + \sigma_t \varepsilon_{t+1}, \qquad \varepsilon_{t+1} \sim \mathcal{N}(0,1), \tag{37}$$

where $\beta^* \equiv \beta g_A^{1-\gamma}$. We solve this stationary model by using the same version of the Smolyak method that is used within EFP to find a solution to T-period stationary economy.

After a solution to the stationary model (35)–(37) is constructed, a solution for nonstationary variables can be recovered by using an inverse transformation $c_t = \widehat{c}_t A_t$ and $k_t = \widehat{k}_t A_t$.

For the sake of our comparison, we also need to recover the path of nonstationary decision functions in terms of their parameters. Let us show how this can be done under polynomial approximation of decision functions. Let us assume that a capital policy function of the stationary model is approximated by complete polynomial of degree L, namely, $\hat{k}_{t+1} = \sum_{l=0}^{L} \sum_{m=0}^{l} b_{m+\frac{(l-1)(l+2)}{2}+1} \hat{k}_t^m z_t^{l-m}$, where b_i is a polynomial coefficient, $i = 0, ..., L+\frac{(L-1)(L+2)}{2}+1$. Given that the stationary and nonstationary solutions are related by $\hat{k}_{t+1} = k_{t+1}/\left(A_0 g_A^{t+1}\right)$, we have

$$k_{t+1} = A_0 g_A^{t+1} \hat{k}_{t+1} = A_0 g_A^{t+1} \sum_{l=0}^{L} \sum_{m=0}^{l} b_{m+\frac{(l-1)(l+2)}{2}+1} \hat{k}_t^m z_t^{l-m} =$$

$$A_0 \sum_{l=0}^{L} \sum_{m=0}^{l} {g_A}^{1-(m-1)t} b_{m+\frac{(l-1)(l+2)}{2}+1} k_t^m z_t^{l-m}. \quad (38)$$

For example, for first-degree polynomial $L = 1$, we construct the coefficients vector of the nonstationary model by premultiplying the coefficient vector $b \equiv (b_0, b_1, b_2)$ of the stationary model by a vector $\left(A_0 g_A^{t+1}, A_0 g_A, A_0 g_A^{t+1}\right)^\top$, which yields $b_{t+1} \equiv \left(b_0 A_0 g_A^{t+1}, b_1 A_0 g_A, b_2 A_0 g_A^{t+1}\right)$, $t = 0, ..., T$, where T is time horizon (length of simulation in the solution procedure). Note that a similar relation will hold even if the growth rate g_A is time variable.

Appendix E. Additional figures

In Figure 10, we plot the simulated solution to the model with both deterministic technology switches and stochastic productivity shocks following an AR(1) process (37);

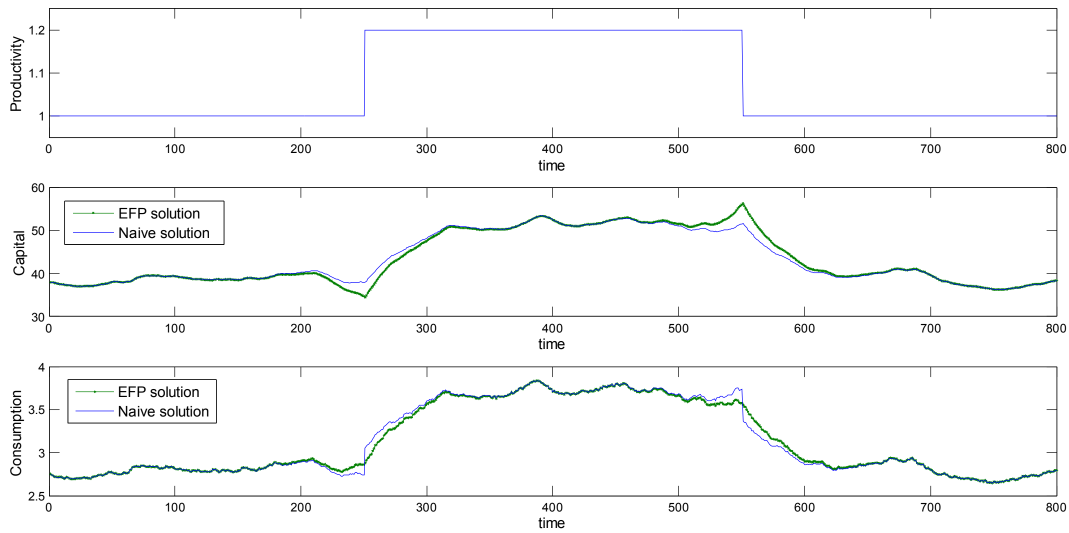

Figure 10. Deterministic technology switches and stochastic productivity shocks

In Figure 11, we provide a plot of simulated solution with both productivity drift and stochastic productivity shocks.

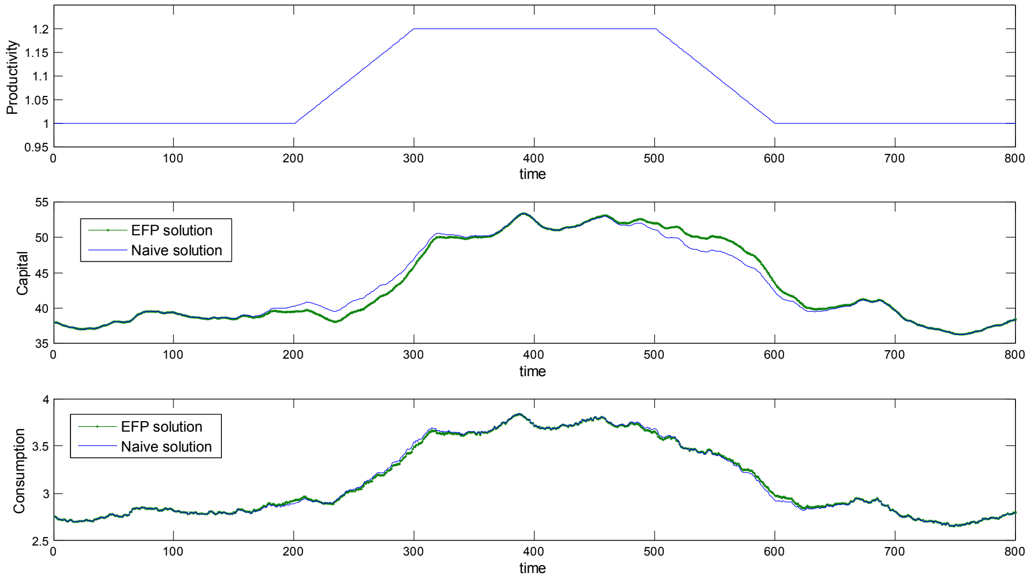

Figure 11. Productivity shifts and stochastic productivity shocks