Optimal Dynamic Stochastic Fiscal Policy with Endogenous Debt Limits

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Questions

- How much can governments use debt to smooth revenues relative to spending?
- What is the debt capacity of an economy?
- How is optimal fiscal policy affected by debt capacity?
- Can we reconcile the size of expected future spending with the low interest rates on government debt?

History

- UK and US use debt to smooth taxation.
  - Borrow when you have a sudden increase in expenditures
  - Good reputation is necessary to access bond market
- Over the past 230 years, the US and UK are the only major economies with constant access to bond markets

UK tax receipts

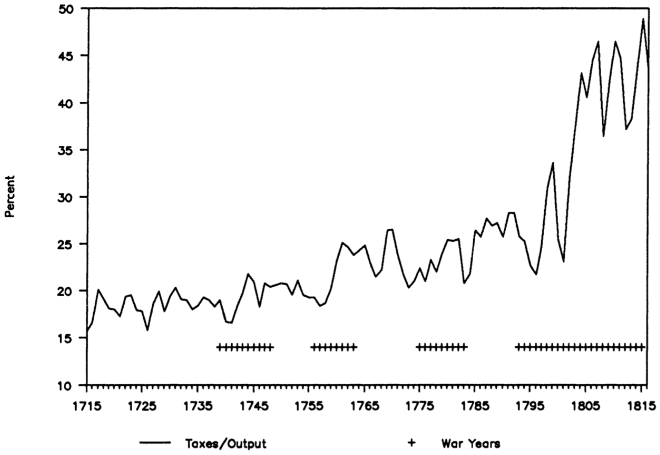


Figure: Bordo and White shows how Great Britain financed the War of the Austrian Succession, Seven Years War, American Revolution, and the wars of the French Revolution and Napoleon.

UK war financing

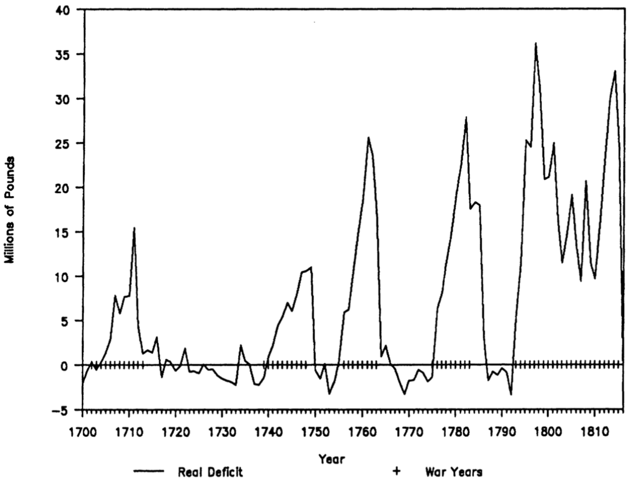


Figure: Bordo and White shows that wars were largely debt financed partially offset by small surpluses in years of peace.

Our Current Motivation: US Debt

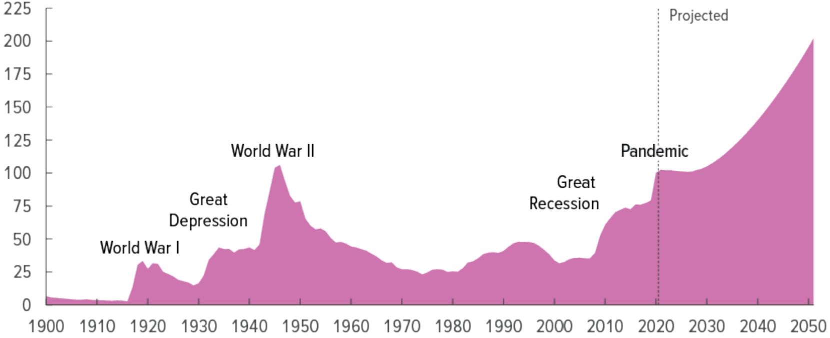


Figure: Federal debt/GDP held by the public, 1900 to 2051.
Data source: Congressional Budget Office.

Scared?

- This ignores unfunded liabilities – Social Security, Medicare, pensions and medical care for Veterans, ...
- Old people like me aren't scared. We have the political power to make sure we get what we were promised, and will be long gone by 2050.
- To young people: If this picture does not scare you, read Larry Kotlikoff's papers

Coverage of this Presentation

- Extension of Barro (1979) (Judd)
- ”Optimal Dynamic Stochastic Fiscal Policy Endogenous Debt Limits” (JMY)
- Comments on AMSS (2002) (Judd)
- Computational methods for solving recursive contracts problems (JMY)

Key Papers

Diamond-Mirrlees (1971) – Paper Zero

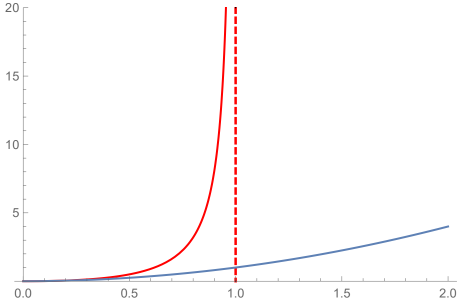
- Assumptions
  - Complete markets
  - Arrow securities make it applicable to dynamic models
  - Arrow securities include ones tied to state of demand for government expenditure
- Implications
  - Thou shalt not tax intermediate goods
  - Tax rates depend on elasticities
  - No reason for correlations between tax rates and contemporaneous spending

Barro (1979)

- Assumptions
  - Exogenous government spending
  - Only asset is a safe bond with a constant interest rate
  - The social cost of tax revenue $\mathcal{T}$ is a convex function, $C(\mathcal{T})$
- Implications
  - If there is no uncertainty over government spending, then taxation is constant over time
  - If there is a one-time permanent change in government expenditure, then taxation jumps to a new and constant level
- Barro Approximation
  - Otimal policy responds to an expenditure shock as if there were no later shocks – certainty equivalent
  - Also the first-order approximation
- Claims
  - Optimal policy implies taxes and debt follow random walks
  - Tax smoothing

Problems with Barro analysis

- Barro's claims hold if $C(\mathcal{T}) = \mathcal{T}^2$
- Quadratic cost function allows unlimited taxation and debt
- Standard public finance models imply a maximum feasible level of taxation; e.g., $C(\mathcal{T}) = \frac{\mathcal{T}^2}{1-\mathcal{T}}$.



Solution to General Barro model

- Assume:
  - Constant rate of interest, r
  - Expected expenditures, G
  - iid spending shocks, z, with variance σ^2 and skewness λ
  - All derivatives of C are negative

Theorem (Judd, 2022)

The third-order expansion for debt dynamics is

$$\mathfrak{D}_{t+1} = \mathfrak{D}_t + z - \sigma^2 \frac{rC'''(r\mathfrak{D}_t + G)}{2C''(r\mathfrak{D}_t + G)} - \lambda \frac{r^2\sigma^3 C''''(r\mathfrak{D}_t + g)}{2C''(r\mathfrak{D}_t + G)} \quad (1)$$

- First-order linearization is a random walk: Barro result
- Negative second-order term proportional to variance
- Negative third-order term if z has positive skewness
- Long run debt is a war chest: $\mathfrak{D}_\infty = -G/r$

Intuition

- The social cost of taxation is infinitely bad at some finite level.
- A long period of high G could with positive (albeit small) probability push debt levels so high that taxes could not cover both expenditures and debt service.
- Optimal policy wants to prevent that possibility.

1980: A Good Year

- Turnovsky-Brock and Kydland-Prescott
  - Treats consumer primal and dual variables as states in the government's problem
  - Applies optimal control to optimal policy problem, as done before in optimal tax literature in the 1970's
- Turnovsky-Brock
  - Applies standard control theory for optimal policies – the "first-order" approach
  - Includes capital, labor, money, and endogenous spending
- Kydland-Prescott
  - Formulates optimal tax problem as a stochastic dynamic programming problem
  - Notes that feasible set of state variables is endogenous, not known independent of the solution

Judd-Müller-Yeltekin Model

Judd-Müller-Yeltekin Assumptions

- Basic assumptions
  - Government can only issue one-period risk-free debt
  - Flat tax rate on labor income
  - Governments can commit to policies
  - Representative agent
  - No capital.
- Key Difference with past models
  - Government spending is endogenous
  - US and UK history shows that WWI and WWII expenditures were chosen, not necessitated by circumstances.

Representative Consumer

- Labor supply is ℓ; time endowment is 1; "leisure" is $1-\ell$
- Utility function (assuming $\theta > 0$, $\eta > 0$, $\underline{c} > 0$)

$$u(c, \ell, g, z) = \frac{(c+\underline{c})^{1-\sigma_1}}{1-\sigma_1} + \eta \frac{(1-\ell)^{1-\sigma_2}}{1-\sigma_2} - \theta(g-z)^2,$$

- z follows a Markov process with transition $\pi(z'|z)$
- Fixed-g case (e.g., AMSS): g is exogenous with $g = z$
- c is market consumption. Marginal utility is finite when $c = 0$ allowing $\ell = 0$ when taxes are high. The revenue-maximizing tax rate is less than 1.

Technology and taxes

- Linear technology for g and c
- Wage and marginal product of labor is $w = 1$
- Aggregate resource constraint: $c + g = \ell$
- Proportional tax τ on labor income
- Transfer is $tr \geq 0$
- We ignore state and local taxation and spending; therefore, the cost of national taxation is underestimated in our model

Risk-free Financial asset

- Non-contingent real one-period gov't bonds, zero net supply
- One bond at t promises one unit of consumption at $t+1$
- b_t - payout of debt at beginning of t
- p_t - price of bond which matures in $t+1$
- $b > 0$ - government is in debt
- $b < 0$ - consumers are in debt

Judd comments on Aiyagari, Marcet, Sargent, Seppala (AMSS) (2002)

Aiyagari, Marcet, Sargent, Seppala (AMSS) (2002)

- The model in AMSS (2002) is the same as JMY but they allow only exogenous expenditures. Below are some comments on that paper that will be collected in a separate paper.
- Micro-based version of Barro (1979)
  - Labor supply, no capital, only safe bonds
  - Exogenous spending
  - IF utility is quasi-linear:
    - Long run debt is negative: government builds a war chest.
    - Long run tax rates are 0.
  - Focuses on martingale properties and asymptotics. However, asymptotically we are all dead.

Comments on MSS Algorithm and Code

- PEA is based on minimizing sum of unconditional errors in dynamic equations. We impose optimality at all times.
- Changes in the initial bond level cause nontrivial changes in solution's coefficients
- Changes in the seed affects the solution
  - Monte-Carlo simulations make solutions random variables
  - MSS does not check the economic importance of this variance
  - Different seeds often lead to non-converging iterations

Comments on MSS Results

AMSS displays a 250-period slice of a simuilation

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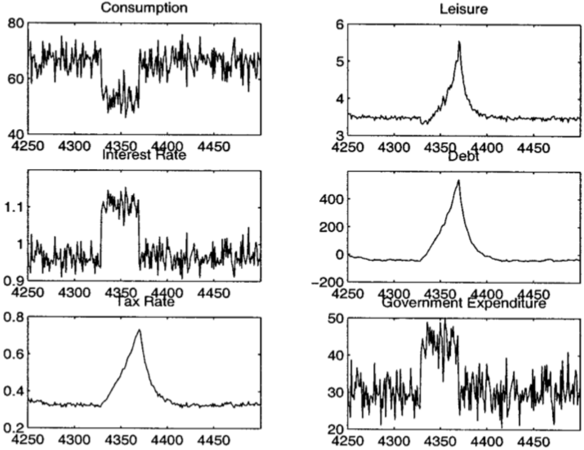


FIG. 7.—Simulation of peace and war economy with incomplete markets

Comments on MSS Results

We ran their code over longer periods we found the following.

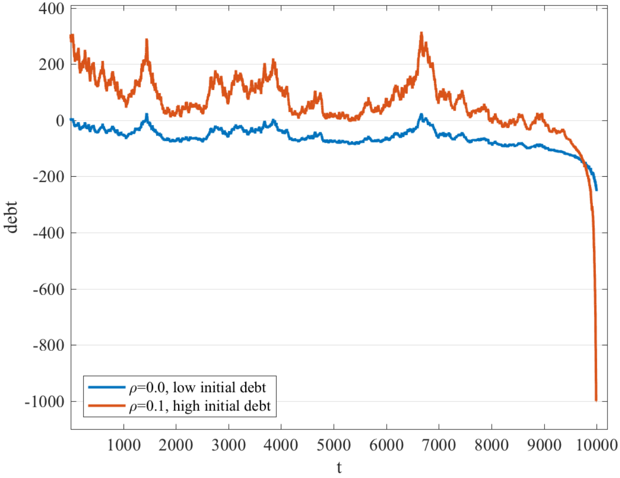


Figure: Nonstationary debt process

Judd-Mueller-Yeltekin analysis

Government: Principal-Agent Narrative

- We follow Kydland-Prescott (1980).
- The current state of the economy is
  - debt b,
  - marginal utility of consumption, λ, and
  - spending state z,
- Gov't chooses
  - today's c, τ, tr, G, b^{+}, p, and
  - a vector of z-contingent values, $\lambda_{+}(z_{+})$, for next period's marginal utility of consumption

  such that all equilibrium constraints are satisfied
- Objective is expected representative agent utility

Computational Challenges

- Value function may not be concave
  - Constraint set may not be convex
  - Problem is an MPCC in general; easy case in our examples

- Global optimization necessary due to multiple local optima

- Must find the feasible sets $\Omega(z)$: The set of all (b, λ) such that there is a bounded process of future (b, λ, z) such that equilibrium conditions surely hold.

- We approximate the feasible set, as in Judd-Yeltekin-Conklin and Yeltekin-Cai-Judd

- We discretize the state space
  - Begin with coarse discretization (200x200, ..., 500x500)
  - Determine the ergodic region
  - Refine grid in ergodic region and resolve Bellman equation

JMY Algorithm

- Uses dynamic programming formulation of optimal tax problem
- Pushes the conditional errors to zero everywhere
- Determines the feasible set of states
- Shows that the dynamic programming approach is quite feasible for dynamic bilevel optimization problems

Equations for Math Nerds

Consumer Problem

- Expected discounted utility

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, \ell_t, g_t, z_t)$$

- Consumer budget constraint:

$$(c_t + p_t b_{t+1}) - (b_t + tr_t + (1 - \tau_t)\ell_t) \leq 0$$

- Consumers choose $\{c_t, \ell_t, b_{t+1}\}$,
- $\lambda_{t+1}(z_{t+1})$ is marginal value of bond next period given z_{t+1}

$$
\begin{aligned}
\text{FOC}c:&\ -\lambda_t^* + uc'(c_t) = 0 \\
\text{FOC}\ell:&\ (1-\tau_t)\lambda_t^* - u\ell'(1-\ell_t) = 0 \\
\text{Euler}:&\ \beta \sum_{z_{t+1}} \lambda_{t+1}(z_{t+1})\pi(z_{t+1}|z_t) - p_t\lambda_t^* = 0 \\
\text{Budget}:&\ -b_t + c_t - tr_t - \ell_t(1-\tau_t) + p_t b_{t+1} = 0 \\
\text{Bounds}:&\ 0 \leq \ell \leq 1,\ c \geq 0
\end{aligned}
$$

Government's Bellman Equation

$$
\begin{aligned}
V(b,\lambda,z) = {} & max\ u(c,\ell,g,z) + \beta EV(b^{+},\lambda^{+}(z^{+}),z^{+}) \\
s.t. \qquad & 0 = c + g - \ell \\
& 0 = -\lambda + uc'(c) \\
& 0 = (1-\tau)\lambda - u\ell'(1-\ell) \\
& 0 = \ \beta \sum_{z^{+}} \lambda^{+}(z^{+})\pi(z^{+}|z) - p\lambda \\
& 0 = \ -b + c - tr - \ell(1-\tau) + pb^{+} \\
& 0 \leq c,\ \ell,\ 1-\ell,\ g,\ \lambda^{+}(z^{+}),\ tr
\end{aligned}
$$

Note: Max over the empty set is $-\infty$.

Therefore, $V(b,\lambda,z) = -\infty$ at infeasible states.

Feasibility:

Definition: A state (b, λ) is feasible if—starting at (b, λ)—there exists a policy series which surely satisfies all constraints at all times.

$\Omega(z)$: The set of all (b, λ) such that it is possible for the government to surely satisfy the dynamic government constraint if current state is (b, λ, z).

Ω: The set of $\Omega(z)$.

Fact: If $(b, \lambda, b^+, \lambda^+)$ satisfy constraints, (b^+, λ^+) are feasible, and

$$b \leq \frac{LS\lambda + b^+\beta\lambda^+ + c(\lambda)(\lambda - u\ell'(1 - c(\lambda)))}{\lambda}$$

then (b, λ) is also feasible.

Useful Fact: With flexible g, feasible set does not depend on z

Algorithm:

1: Find a set of feasible state, Ω (e.g., war chest states)

2: Find (b, λ) such that some (b^+, λ^+) in Ω makes (b, λ) feasible and add them to Ω

3: Repeat until convergence

Value Function Approximation

- Value function is singular at boundary of feasible region
- Few shape properties available to exploit
- Discrete state solution may lead to a continuous approximation
  - Discrete state solution reveals singularities at boundary
  - Singularity information implies continuous approximations

Government's Bellman Equation

For $X = (b, \lambda, z, b^{+}, \lambda^{+}(z^{+}))$ define:

$$
\begin{aligned}
U(X) \equiv \max_{c,\ell,p,g,tr,\tau} \quad & u(c, \ell, g, z) \\
s.t. \quad & 0 = c + g - \ell \\
& 0 = -\lambda + uc'(c) \\
& 0 = (1-\tau)\lambda - u\ell'(1-\ell) \\
& 0 = \beta \sum_{z^{+}} \lambda^{+}(z^{+})\pi(z^{+}|z) - p\lambda \\
& 0 = -b + c - tr - \ell(1-\tau) + pb^{+} \\
& 0 \leq c,\ \ell, (1-\ell),\ \ g,\ \lambda^{+}(z^{+}),\ \ tr
\end{aligned}
$$

- At each (b, λ, z) and $X = (b, \lambda, z, b^+, \lambda^+(z^+))$,

$$V(b, \lambda, z) = \max_{b^+, \lambda^+(z^+)} U(X) + \beta EV(b^+, \lambda^+(z^+), z^+)$$

$$s.t. \quad (b^+, \lambda^+(z^+)) \in \Omega$$

- Use policy function iteration.
- Global optimization means we need to compute $U(X)$ for all feasible future states.
- Requires excellent constrained optimization software.
- Algorithm keeps track of X combinations that are not feasible.

- Example with Pre-computation of $U(X)$
  - If utility function is log for both uc and ul, then $U(X)$ has a closed-form solution
  - The precomputed $U(X)$ reduces the cost of each iteration
- Work-in-Progress: Pre-computation of $U(X)$ for all cases
  - Perhaps $U(X)$ can be approximated; always possible for fixed-g case.
  - Cost of computing $U(X)$ may be high, but precomputing $U(X)$ reduces cost of each iteration (IRTS)
  - Approximation must be excellent to achieve acceptable precision in Bellman equation

Computational Resources: Closed-form $U(X)$

Grid size	200 x 200	300 x 300	400 x 400	500 x 500
Total time [h]	0.277	1.1894	8.005	29.803
Time/step [h]	0.016	0.118	0.500	1.480

- Hardware: One CPU With NVIDIA Tesla P100 GPU
- Closed-form $U(X)$
- Global optimization strategy: Global search in Ω
- Implication: Multi-CPU/Multi-GPU architecture with precomputed $U(X)$ will easily solve far more complex extensions

General Results

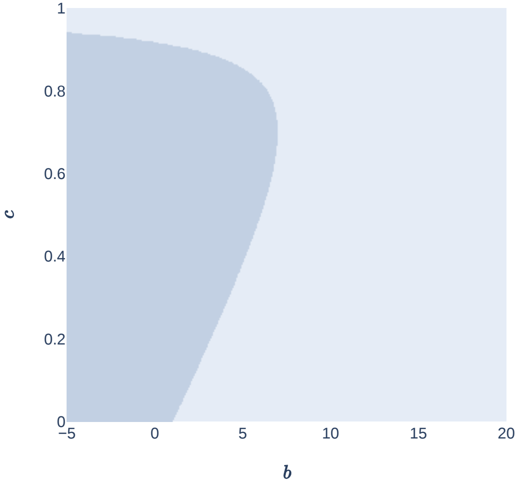
Feasibility

Two Cases: Flexible g vs. Fixed g

- Log - log utility function
- Feasible set is an analytical constraint on DP problem.
- Two z shocks: low spending (peace) and high spending (WWII for US).

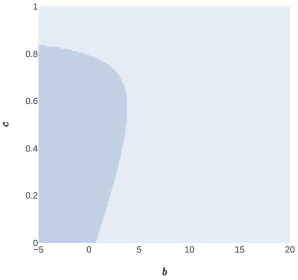
Case	σ_1	σ_2	$\underline{c}$	g	g shocks	g/GDP shocks
Flexible	1.0	1.0	0.1	flex	$\{0.09, 0.27\}$	$\{16\%, 40\%\}$
Fixed	1.0	1.0	0.1	fixed	$\{0.09, 0.27\}$	$\{16\%, 40\%\}$

Feasible Set, Flexible g

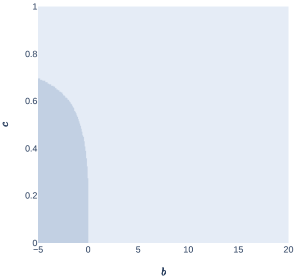


Feasible set does NOT depend on preferences over g.

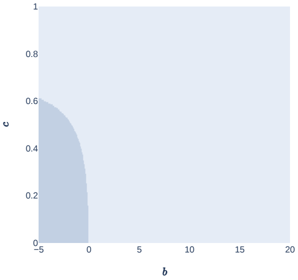
Feasible Set, Fixed g



(a) War = 16% GDP



(b) War = 28% GDP



(c) War = 40% GDP

General Result: As size if war increases, the feasible level of debt drops and often does not allow any positive level.

An Illustrative Example with Flexible G

Parameters for an example

- $\beta = 0.96$, $\underline{c} = 0.1$
- Utility function:

$$u(c, \ell, g, z) = \log(c + \underline{c}) + \log(1 - \ell) - 100(g - z)^2.$$

- Transition probability matrix

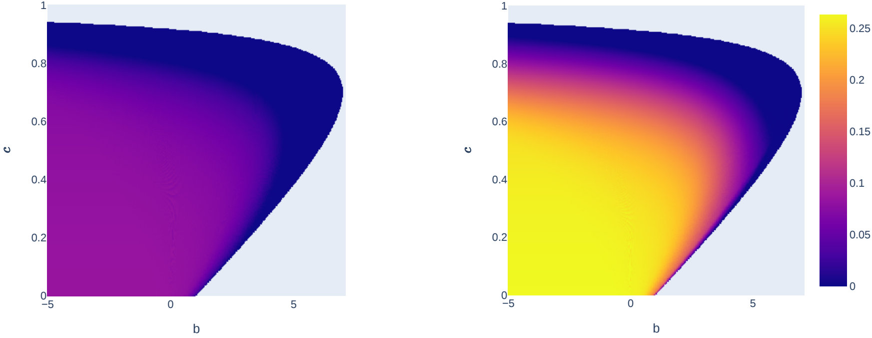
$$\Pi = \begin{bmatrix} 0.9787 & 0.0213 \\ 0.3333 & 0.6667 \end{bmatrix}; \tag{2}$$

  roughly two major wars per century with an average duration of 3 years

- z shocks = $\{0.09, 0.27\}$

Contour Plots

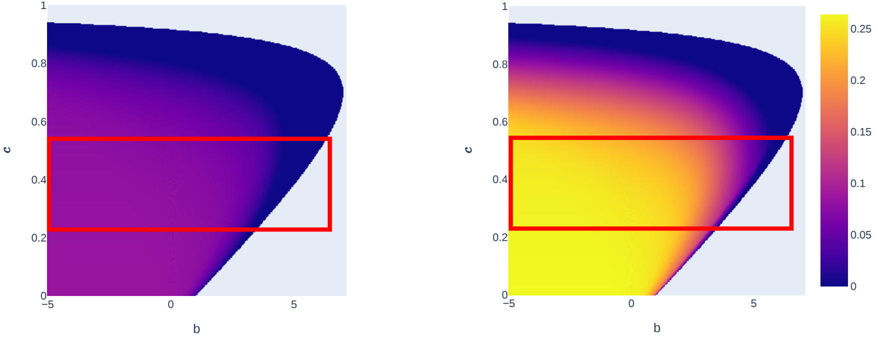
Full region, Government Spending



(a) Peace (b) War

G equals target level except for nearly infeasible levels of debt, where G drops to zero.

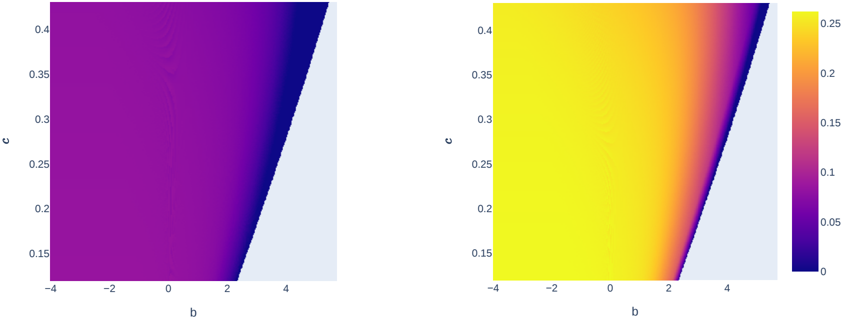
Full region, Government Spending



(a) Peace (b) War

After a few periods, the state is always in the red box, which we call the ergodic box.

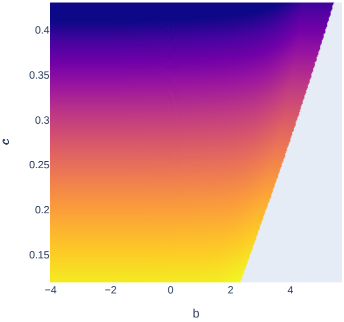
Ergodic Region, Government Spending



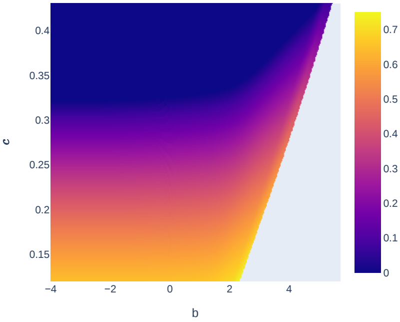
(a) Peace

(b) War

Ergodic Region, Tax Rate



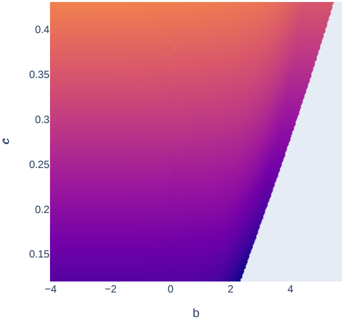
(a) Peace



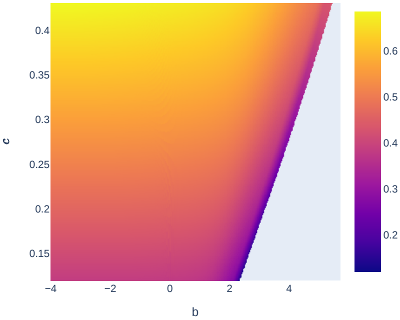
(b) War

Tax rate is low except with when debt is high, where tax rate approaches max revenue rate.

Ergodic Region, Labour



(a) Peace

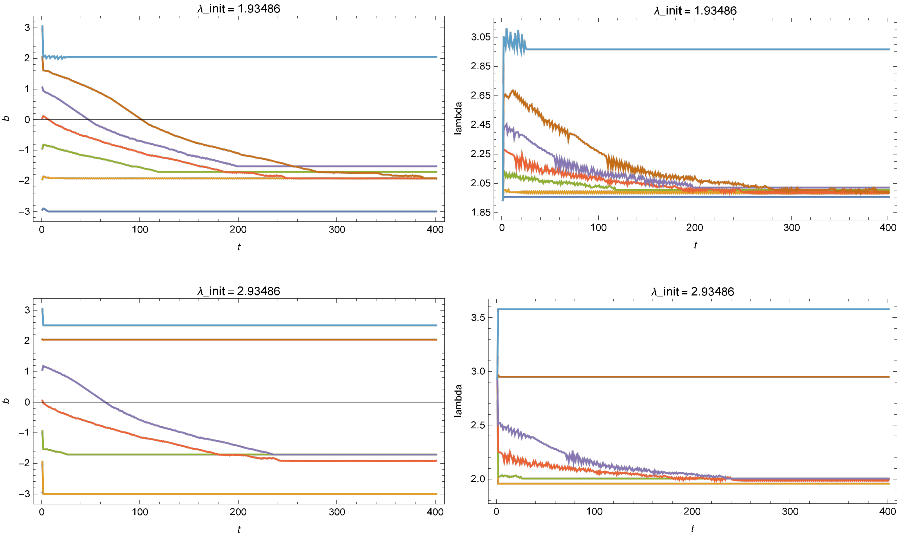


(b) War

Labor is close to first-best except when debt is high, where tax rate is high.

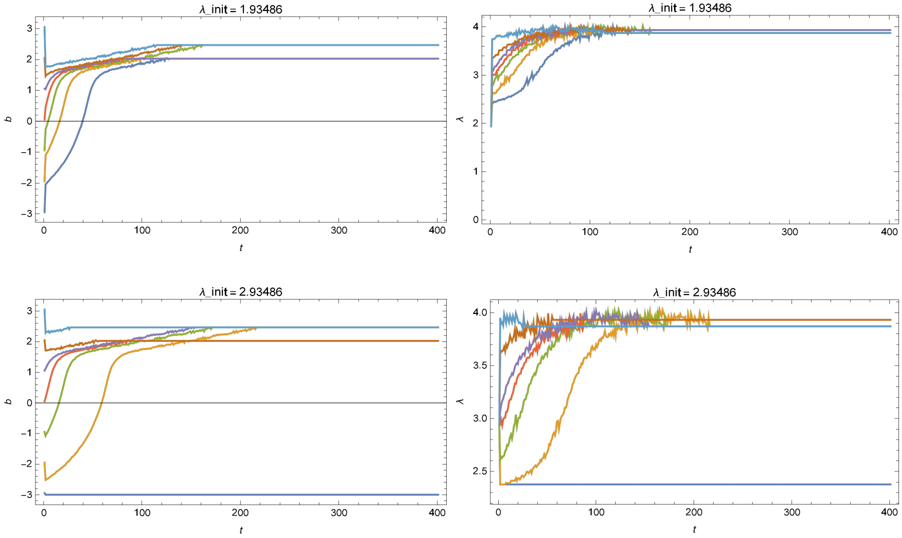
Dynamics

Perpetual Peace



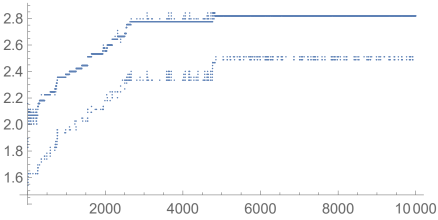
Multiple absorbing states.

Perpetual War

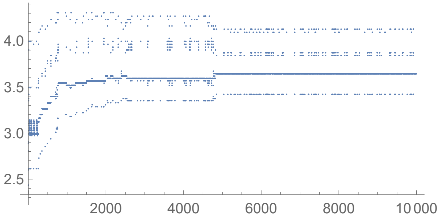


Multiple absorbing states.

Example Time Series Plots



(a) b



(b) λ

Computational Resources: General Case

Grid size	200 x 200
Total time [h]	15.48
Average time/step [h]	0.91

- Hardware: 2 Intel Xeon CPU E5-2698 (40 cores)
- Implication: will be easy to solve extensions (e.g., capital, business cycle shocks, etc.) on 2022 hardware.
  - Multiple places where precomputation can improve efficiency
  - Cai-Judd-Lontzek (JPE, 2019) used 80,000 cores (in 2014) for some examples
  - Yeltekin-Cai-Judd used 160,000 cores (in 2014) in some test runs to compute all Nash equilibria

Summary

Judd-Müller-Yeltekin Results

1. No tendency to accumulate a war chest
2. Long-run may be high debt and taxes, low spending
3. Very slow dynamics at all levels of debt
4. If spending is exogenous, debt may be infeasible
5. If spending is endogenous, feasibility does not depend on preferences over g
6. Making government spending endogenous drastically changes an economy's debt capacity; essential to any plausible analysis of fiscal policy

Potential Next Steps for This Work

- Meet minimal documentation standards
  - Verification: Check that optimality conditions hold
  - Uncertainty Quantification: Construct a map from parameters to properties of solution; see Cai-Judd
  - Post solutions and code for verification and simulations, as in Cai-Judd-Lontzek (JPE, 2019).
  - Open source is not required; often impossible
- Port to third millenium computers
  - Exploit asynchronous parallelization
  - Acquire computer time: Aurora? Add nuclear war and apply for time on Sierra? or a Chinese supercomputer?

Feasible - and Tractable – Extensions

- Capital, business cycle shocks
- Heterogeneous agents (using Judd-Maliar-Maliar papers, GSSA and EDS)
- Heterogeneous agents (using Cai-Judd papers, NLCEQ and SCEQ)
- Incorporate recent advances in principal-agent problems (Renner-Schmedders papers using algebraic geometry methods)
- Redistributive taxation (using Ma-Judd-Orban-Saunders)
- Nash equilibrium of legislative process (using Yeltekin-Cai-Judd).

Conclusion

- Dynamic principal-agent optimization problems can be solved with dynamic programming
- Must use advance tools, such as algebraic geometry, approximation theory, the best solvers, parallelization, and third millenium hardware – just like all fields other than economics are.